GENERALIZED FORMS OF REYNOLDS EQUATION AND THEIR MATHEMATICAL ANALYSIS

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By

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TO

My Mother

CERTIFICATE

This is to certify that the matter embodied in the thesis entitled "Generalized forms of Reynolds equation and their mathematical analysis" by Mr. Suthash Kumar for the award of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by him under my supervision and guidance. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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CONTENTS

CHAPTER - I

GENERAL INTRODUCTION

1.1 INTRODUCTION:

Though the use of lubricant in relatively moving parts to facilitate motion is known to man kind since the invention of wheel and axle, it was only in 1886 that the real mechanism of lubrication was explained by Reynolds [1886] in his classical paper. The object of lubrication in mechanical system is to minimize wear and energy loses by reducing friction between the two relatively moving surfaces (plane/curved) by the presence of an interposed film of lubricant. The proper knowledge and understanding of the lubrication process are essential in improving the standard of design and the efficiency of the system. The characteristics of the bearing system depend upon the following types of lubrication process.

1. Hydrodynamic lubrication
2. Boundary lubrication
3. Mixed lubrication.

In hydrodynamic lubrication, the sliding surfaces are separated by a layer of fluid-film of thickness of the order $10^{-5} - 10^{-3}$ cm and the load capacity and the friction force depend upon the viscosity of the lubricant and the geometry of the film, Pinkus and Sternlicht [1961].

In boundary lubrication, film-thickness between the two surfaces is very small (of the order of a molecular thickness) and the friction is determined by the physical and chemical properties of the film and the solids in contact. Boundary lubrication is assoaiated with the effect of adsorbed hydrocarbon molecules which seperate the two surfaces and the real contact takes place along the 'tails' of molecules sticking out of the surfaces, Bowden and Tabor [1950, 1964] and Ling et. al. [1969].

In the case of mixed lubrication, a situation between hydrodynamic and boundary lubrications, the fluid film between the two sliding surfaces is thin enough and the surface asperities begin to interfere in the hydrodynamic process. Here the load is partly supported by asperity contacts and partly by fluid film. The force of friction also depends upon the interaction of the surface asperities and the hydrodynamic action of the film, Fuller [1954], Lenning [1960], Dobry [1964], Christensen [1972] and Tsao and Tong [1975].

In general, the lubrication characteristics of the bearing system depend upon the following:

(1) Nature of lubricant: liquid or gas, its Newtonian or non-Newtonian behaviour, variation of viscosity or consistency with temperature, pressure or concentration which may change along as well as across the film-thickness, its affinity with surface such as adsorption, absorption, chemical reaction etc., Braithwaite [1967].

(2) Nature of surface: surface roughness, elasticity, thermal conductivity, hardness, porosity etc., its affinity with the lubricant.

(3) Effects of flow regimes , bearing configuration, film-thickness, boundary conditions and the surrounding environment etc.

To study the behaviour of any lubricated system a mathematical model is made by taking the above factors into account which depend upon a given physical situation. The bearing characteristics such as load, flow flux, friction force etc. depend upon the pressure generated in the film and the lubrication process. The survey of various attempts, made by many workers to obtain the governing equation for pressure in the lubricant film, are summarized here.

1.2 SURVEY OF REYNOLDS EQUATION

In the case of hydrodynamic lubrication, the equation governing the fluid film pressure in a bearing can be obtained by coupling the equations of motion and the equation of continuity. This equation was first derived by Reynolds [1886] in his classical paper by considering the following assumptions:

(i) The radious of curvature of the bearing surface is large in comparison to the film-thickness.

(ii) The lubricant is an incompressible Newtonian fluid.

(iii) The lubricant viscosity is constant.

(iv) Inertia and body force terms are small incomparison to the viscous and pressure terms.

(v) Owing to the smallness of the fluid film-thickness, the velocity gradients across the film are large incomparison to the velocity gradients along the film.

(vi) There is no slip at the fluid and solid interface.

Since early fourties, various attempts have been made to generalize the Reynolds equation applicable to bearing system functioning under unusual conditions such as high temperature, high pressure etc.

The first attempt in this direction was made by Fogg [1946] who proposed the thermal-wedge concept in the lubricant film. Cope [1949] relaxed the assumption made by Reynolds [1886] to extend the theory for variable viscosity and density along the film. But the variations of fluid pressure and fluid properties across the fluid film were still neglected. In [1950], Wannier has shown that the basic Reynolds equation can be derived from the equations of motion by considering the variation of fluid pressure across the film. Later, Halton [1958] extended the investigation performed by Cope [1949] and derived a generalized form of Reynolds equation for rough bearings. Several other studies have also been conducted by considering the temperature and viscosity variations along as well as across the film, Zienkiewicz [1957], Cameron [1958], Cameron and Wood [1958], Hunter and Zienkiewicz [1960], and Young [1962].

In early sixties, an unified approach was made by Dowson [1962] to generalize the Reynolds equation by considering the variation of fluid properties across as well as along the lubricant film. Dowson and Hudson [1963-a, 1963-b] , Curie et. al. [1965] , Hahn and Kettleborough [1967] , McCallion et. al. [1970] have also studied the effect of viscosity variation in lubricated systems under isothermal or adiabatic conditions by using numerical techniques. Recently, the performance of viscosity variation across as well as along the hydrodynamically lubricated film was studied by Quale and Wiltshire [1972] , Ezzat and Rohde [1973-a, 1973-b] .

A different approach to study the effect of viscosity variation has also been proposed by Tipei [1962] , Tipei and Nica [1967] and Tipei and Degueurce [1974] by assuming a relationship between viscosity and film-thickness for convergent films.

SLIP EFFECTS:

In the above mentioned studies, one of the basis assumptions used is no-slip condition at the bearing surface. However, in general this assumption is not valid and the effect of slip may be important on the flow behaviours of gases and liquids especially when the fluid film-thickness is very small, Kennard [1938] , Schaaf and Chambre [1958] ; the surface is very smooth, Devenport [1973] ; and when the bearing surface is porous, Beavers and Joseph [1967] . In the case of gas lubricated bearings, as the density is lowered,

the molecular mean free path becomes comparable to the film-thickness, the gas seems to lose its grip upon the surface and leaves it with a finite slip-velocity, Kennard [1938] , Grad [1949] and Schaaf and Chambre [1958] .

Burgdorfer [1959] was the first to derive a modified form of Reynolds equation applicable to hydrodynamically lubricated gas bearings with slip-flow condition, Gross [1962]. Hsing and Malanoski [1969] have studied the effect of molecular mean free path in the spiral-grooved thrust bearing by considering the slip-velocity at the surface. The rarefaction effects of gas lubricated bearings in magnetic recording disk file has also been studied, Tseng [1975] .

In the case of liquids, the slip flow can be important when the surface is smooth and bearing is functioning at high temperature, Lamb [1945] , Bird [1956] and Devenport [1973] . Slippage at the wall is more in the case of high temperature where the viscosity of the base oil decreases near the surface. This effects may be eliminated by roughening the surface, Bramhall and Hutton [1960] and Devenport [1973] .

The existance of slip flow at the porous boundary also been demonstrated by Beavers et. al [1967, 1970] and it has further been supported by Saffman [1971] and Taylor [1971] . The effect of slip flow in porous bearings has been studied by Sparrow et.al [1972] , Wu [1972] , Jones [1973] , Murti [1973] and Beavers et. al.[1974] .

In general, the effect of slip is to decrease the load capacity of the bearing and the friction force at the bearing surface.

ROUGHNESS EFFECTS

As it has been pointed out earlier, the topology of the bearing surface plays an important role in the lubrication mechanism. In one of the approaches, the effect of surface roughness is taken into account in the usual Reynolds equation by considering that the film-thickness is a function of surface roughness which may be represented by a series of sine or cosine waves, Burton [1963] , Shukla and Prasad [1966] . This procedure has also been applied to study the bearing characteristic of rollers, Dowson and Whomes [1971] and of spiral groove bearing, Wildmann [1968] .

In another method, the surface roughness is assumed to be represented by a stochastic process and the usual procedure is followed by taking statistical mean or average of the basic governing equation, Papoulis [1965] . Using thie approach, the characteristics of an infinite slider bearing and short journal bearings have been studied by considering the distribution of the surface roughness to be gaussian, Tzeng and Saibel [1967-a, 1967-b] . Using this approach, Christensen [1969-70] has derived the generalized form of Reynolds equations applicable to finite rough bearings and its various particular cases have been investigated, Christensen [1971] , Christensen and Tonder [1969,1971

Tonder and Christensen [1972-a, 1972-b] . Another form of Reynolds equation applicable to rough or deformed surfaces was presented by Berthe and Godet [1973] . A further refinement of this theory has been made recently by Christensen et.al. [1975] . The concept of stochastic approach has also been extended to rough bearings using compressible lubricants, Elord [1973] , Christensen, et. al. [1975] .

1.3 MIXED LUBRICATION

In the above, a survey of Reynolds equation applicable hydrodynamically lubricated rough bearings where the asperities heights are very small incomparison to the norminal film thickness has been presented. However, when the heights of the surface asperities are of the same order of magnitude as the film-thickness the direct contact between the opposite surfaces may occur, causing the condition of mixed lubrication. In this case, the total load may be partly supported by the asperities contact and partly by hydrodynamic action. Similarly, the total friction force will arrise partly from boundary lubrication condition associated with the asperities contact and partly from hydrodynamic action, Fuller [1954] , Lenning [1960] and Dobry [1964] . Christensen [1969] and Fowles [1971] have studied the asperities deformation and hydrodynamic effects in lubricated contact. The transition from boundary to mixed lubrication has also been investigated by Thomson and William [1972] . Recently, a theory of mixed lubrication by using stochastic Reynolds equation

has been presented by Christensen [1972]. Tsao and Tong [1975] have also studied the condition of mixed lubrication by calculating the roughness interference area from the knowledge of bearing geometory and surface roughness.

Keeping the above in view, in this thesis an attempt has been made to give new ideas and to generalize some of the older theories presented before in the theory of lubrication. The following summary gives the details of the work done presented in this thesis.

1.4 SUMMARY

This thesis consists of five chapters. Chapter-I deals with the general introduction of the literature related to the work presented in the thesis.

In Chapter-II, a generalized form of Reynolds equation with slip at the bearing surfaces is derived by considering the variation of fluid properties across as well as along the film. For gas-lubricated bearings,a generalized form of Reynolds equation is deduced in terms of molecular mean free path. The case of hydrostatic bearing is studied and it has been shown that the load capacity decreases as molecular mean free path increases.

In the case of liquids, a generalized form of Reynolds equation is derived by considering the viscosity of the lubricant as a step

function across the film in multiple layer lubrication. The case of slider and hydrostatic bearings are studied. It has been shown that the load capacity, friction force increase as the coefficient of sliding friction and viscosities of layers increase.

Further, in the case of porous bearing, a generalized form of Reynolds equation is deduced by assuming the existance of slip flow at the porous surface. Moreover, the c se of one dimensional externally pressurized porous squeeze-film bearing has been investigated.

In Chapter-III, by using the theory of stochastic process, a generalized form of Reynolds equation applicable to rough bearings is derived by assuming that the fluxes are represented by power series of stochastic film-thickness function. The case of hydrodynamic one-dimensional step bearing and hydrostatic bearing are investigated. In the case of a step bearing, it has been shown that the load capacity and friction force increase but the coefficient of friction decreases as the roughness parameter increases. Similar results are true in the case of a hydrostatic bearing.

Chapter-IV deals with the generalization of the work presented in Chapter-III, by (considering the variation of viscosity along as well as across the film-thickness and stochastic form of Reynolds equation has been derived. The effects of viscosity variation in the cases of slider and hydrostatic bearings have been studied.

In Chapter-V, a new theory of mixed lubrication is presented by considering the surface asperities to be randomly distributed fins or cones which peneratrate through the fluid film and partly may be in contact with the asperities of the opposite surface. They are considered to form a net-work of interacting zone corresponding to each surface through which the lubricant flows as happens in the case of flow through porous channels with singular resistances or some what as in the case of flow through porous matrix.

A generalized form of Reynolds equation is derived by considering a modified forms of Navier-Stoke's equations in these interacting zones. As particular cases, step slider bearing and hydrostatic step bearing have been studied. It has been shown that the load capacity and friction force increase as the interference to the flow increases or as viscosities of layers increase.

This theory has also been applied to the study of synovial joint. It has been pointed out that the load capacity and time of squeezing decrease in the case of fractured joint under impact loading condition. However, this may be compensated by boosting effects which increases the viscosity of the synovial fluid.

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CHAPTER - 2

EFFECTS OF SLIP AND VISCOSITY VARIATION IN HYDRODYNAMIC LUBRICATION

2.1 INTRODUCTION

In general, most of the lubricated systems can be considered to consist of moving (stationary) surfaces (plane/curve, loaded/unloaded) with a thin film of an external meterial (lubricant) between them. The presence of such a thin film between these surfaces not only helps to support considerable load but also minimizes friction. The characteristics, such as pressure in the film, frictional force at the surface, flow rate of lubricant etc. of the system depend the nature of the surfaces, the nature of the lubricant film, boundary conditions etc.

The equation governing the pressure generated in the lubricant film can be obtained by coupling the equations of motion and the equation of continuity. This equation was first derived by Reynolds [1866] in his classical paper under usual lubrication assumptions for an incompressible lubricant and is now known as 'Reynolds equation'. In deriving this equation, the thermal and compressibility effects as well as viscosity variation in the film were ignored. Later Cope [1949] modified the Reynolds equation by including viscosity and density variation along the fluid film. The viscosity variation across the

film thickness has been considered by Zienkiewicz [1951,1957], Cameron [1958] who also pointed out that temperature gradients and viscosity variation across the film may not be ignored. In the year [1962], Dowson unified the various attempts in generalizing the Reynolds equation by considering the variation of fluid properties across as well as along the fluid film-thickness by neglecting slip effects at the bearing surfaces.

The effects of viscosity variation have also been considered by Quale and Wiltshire [1972] and Tipei and Degueurce [1974] without considering slip at the surfaces. However, it has been noted that the effect of slip may be important on the flow behaviour of gases and liquids particularly when the film-thickness is very small, Kennard [1938], Grad [1949] and Gross [1962] and the surface is very smooth, Devenport [1973]. The effect of slip can also be important at the porous boundary, Beavers and Joseph [1967].

In the gas bearing applications, such as gyroscope, Heinrich [1970], the bearings are essentially operated in the slip-flow regime. This happens because of either low pressure or extremely thin bearing film when the length of the molecular mean free path λ becomes comparable to the film-thickness h, Kennard [1938] and Grad [1949]. For smaller, but non-negligible value of λ/h, the immediate adjacent layer of the gas to the solid surface is no longer attached to the surface but has a finite relative slip velocity and this produces the effect of an apparent diminuation in the viscosity of the gas.

To study the effect of slip, Burgdorfer [1959] modified Reynolds equation for gas-lubricated hydrodynamic bearings with "slip flow" under isothermal condition. He pointed out if $0 < \frac{\lambda}{h} < 1$, gas flow may be considered to be continuous and analysis can be carried out with modified slip boundary conditions. Also Hsing and Malanoski [1969] studied the effect of molecular mean free path in the spiral-grooved thrust bearing and Tsing [1975] used the Reynolds equation to study the rarefaction effects of gas-lubricated bearings in magnetic recording disk file.

Slip flow boundary condition at the wall in the case of gas bearing can be written as Kennard [1938] and Burgdorfor [1959],

$$u_{slip} = \sigma\left(\frac{2-f}{f}\right) \lambda\left(\frac{\partial u}{\partial z}\right)_{wall} + \cdots \tag{2.1}$$

where f is reflection coefficient, λ is the mean free path and σ is numerical constant. Because of σ and f are closed to unity, it may be considered that $\sigma\left(\frac{2-f}{f}\right)$ is unity. As Molecular mean free path λ depends upon the fluid viscosity, pressure and temperature it can be approximated by the following relation, Tsing [1975],

$$\lambda \simeq \frac{16}{5\sqrt{2\pi}} \frac{\eta}{p} \sqrt{RT} \tag{2.2}$$

where R is a gas constant and T is the temperature, η is the viscosity of the gas and p is its pressure.

The effect of wall slippage in the case of liquids with dispersed material and greases have been discussed by Devenport [1973]. Slippage may be more at higher temperature when the base oil viscosity reduces but this may be eliminated by roughening the solid boundary. The magnitude of slip depends upon the type of fluid and surface roughness. Mathematically, the effect of slip has been pointed out by Lamb [1945], Bird [1956] and Bramhall and Hutton [1960]. The velocity of slip at the wall in the case of liquid, can be defined as follows:

$$u_{slip} = \frac{1}{\beta} \left(\eta \frac{\partial u}{\partial z}\right)_{wall} \tag{2.3}$$

where β is the coefficient of sliding friction at the wall.

The slip flow is not only important at the solid bearing surface but it plays an important role in bearings with porous facing. This effect has been discussed by Beavers, et. al. [1967, 1970] for an incompressible fluid in their experiment who demonstrated the existance of slip-velocity at the porous surface and proposed an emperical boundary condition. The work was further supported by Taylor [1971], Saffman [1971] by providing theoretical justifications.

Effect of slip flow in the case of porous squeze film and porous spherical shell were considered by Sparrow, et. al. [1972], Wu [1972], Jones [1973], and Murti [1973]. Beavers et. al. [1974] have also proposed a boundary condition at a porous surface

and experiments were performed to establish the slip flow for porous gas bearing. The slip velocity at the porous surface can be written as, Beavers et. al. [1974]

$$u_{slip} = \frac{(\phi)^{1/2}}{\xi} \left(\frac{\partial u}{\partial z}\right)_{wall} \tag{2.4}$$

where ξ is the slip coefficient at the wall and ϕ is the permeability of the porous facing.

In view of the above, in this chapter, a generalized form of Reynolds equation for fluid film lubrication with slip velocity at the surface is derived, by considering the variation of fluid properties across as well as along the film-thickness, and various particular cases are studied.

2.2 BASIC EQUATIONS:

Consider the laminar flow of a fluid between two surfaces whose physical configuration is shown in Fig. (2.1). Considering the variation of fluid properties across as well as along the film-thickness, the basic equation of motion and equation of continuity in their most general form for a Newtonian fluid can be written as follows:

$$\rho \frac{Du}{Dt} = \rho X - \frac{\partial p}{\partial x} + \frac{2}{3} \frac{\partial}{\partial x} \eta\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{2}{3} \frac{\partial}{\partial x} \eta\left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)$$

$$+ \frac{\partial}{\partial y} \eta\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z} \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

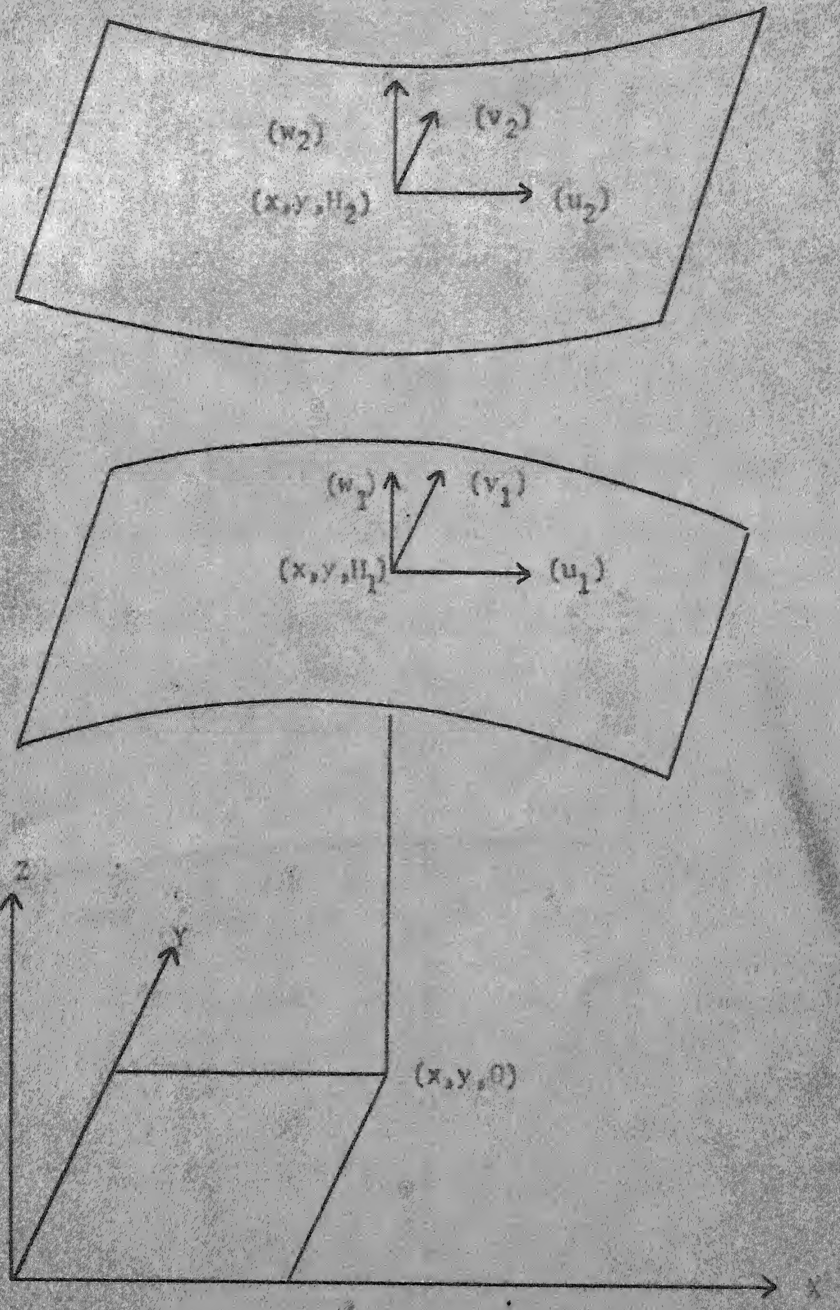

Fig. 2.1 COORDINATE SYSTEM

$$\rho \frac{Dv}{Dt} = \rho Y - \frac{\partial p}{\partial y} + \frac{2}{3} \frac{\partial}{\partial y} \eta\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) + \frac{2}{3} \frac{\partial}{\partial y} \eta\left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)$$

$$+ \frac{\partial}{\partial z} \eta\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial x} \eta\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad (2.5)$$

$$\rho \frac{Dw}{Dt} = \rho Z - \frac{\partial p}{\partial z} + \frac{2}{3} \frac{\partial}{\partial z} \eta\left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\right) + \frac{2}{3} \frac{\partial}{\partial z} \eta\left(\frac{\partial w}{\partial z} - \frac{\partial v}{\partial y}\right)$$

$$+ \frac{\partial}{\partial x} \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y} \eta\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (2.6)$$

To derive the generalized form of Reynolds equation with the usual assumptions of lubrication theory are made, Dowson [1962]. Following him, the equations (2.5) could be simplified as follow:

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(\eta \frac{\partial u}{\partial z}\right) \qquad (2.7)$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial z}\left(\eta \frac{\partial v}{\partial z}\right)$$

where $p = p(x,y)$ is the pressure in the film.

Considering the slip at the surfaces, the boundary conditions can be written as follows:

$$u = (u)_1 = \lambda_1\left(\frac{\partial u}{\partial z}\right)_1 + U_1\,, \quad z = H_1$$

$$v = (v)_1 = \delta_1\left(\frac{\partial v}{\partial z}\right)_1 + V_1\,, \quad z = H_1 \qquad (2.8)$$

$$u = (u)_2 = -\lambda_2\left(\frac{\partial u}{\partial z}\right)_2 + U_2\,, \quad z = H_2$$

$$v = (v)_2 = -\delta_2\left(\frac{\partial v}{\partial z}\right)_2 + V_2\,, \quad z = H_2$$

where λ's and δ's are molecular mean free path for gas lubrication, and depend upon lubricant temperature, pressure and viscosity. In the case of liquids they depend upon viscosity and the coefficient of sliding friction. However, in the case of porous bearing, these are functions of slip coefficient at the wall and permeability parameter of porous facing, [see equations (2.1) to (2.4)] .

Integrating equation (2.7) and using boundary conditions (2.8), the following expressions for the fluid film velocities are obtained.

$$u = U_1 + [\alpha_1 H_1 \int_{H_1}^{z} \frac{zdz}{\eta}] \frac{\partial p}{\partial x} + [\frac{(U_2-U_1)}{F_o} - \frac{F_1}{F_o}\frac{\partial p}{\partial x}] [\alpha_1 + \int_{H_1}^{z} \frac{dz}{\eta}]$$

$$v = V_1 + [\beta_1 H_1 + \int_{H_1}^{z} \frac{zdz}{\eta}] \frac{\partial p}{\partial y} + [\frac{(V_2-V_1)}{F'_o} - \frac{F'_1}{F'_o}\frac{\partial p}{\partial y}] [\beta_1 + \int_{H_1}^{z} \frac{dz}{\eta}] \tag{2.9}$$

where

$$F_o = \alpha_1 + \alpha_2 + \int_{H_1}^{H_2} \frac{dz}{\eta}$$

$$F'_o = \beta_1 + \beta_2 + \int_{H_1}^{H_2} \frac{dz}{\eta}$$

$$F_1 = \alpha_1 H_1 + \alpha_2 H_2 + \int_{H_1}^{H_2} \frac{z\,dz}{\eta} \tag{2.10}$$

$$F'_1 = \beta_1 H_1 + \beta_2 H_2 + \int_{H_1}^{H_2} \frac{zdz}{\eta}$$

$$\alpha_1 = \frac{\lambda_1}{\eta_1} \; , \; \alpha_2 = \frac{\lambda_2}{\lambda_2}$$

$$\beta_1 = \frac{\delta_1}{\eta_1} \; , \; \beta_2 = \frac{\delta_2}{\eta_2} \; .$$

Integrating equation of continuity (2.6) with respect to z and taking limits from $z = H_1$ to $z = H_2$, we get

$$\int_{H_1}^{H_2} \frac{\partial \rho}{\partial t}\, dz + \int_{H_1}^{H_2} \frac{\partial}{\partial x} (\rho u)\, dz + \int_{H_1}^{H_2} \frac{\partial}{\partial y} (\rho v)\, dz + [\rho w]_{H_1}^{H_2} = 0 \tag{2.11}$$

Applying the well known integral formula

$$\frac{\partial}{\partial x} \int_{H_1}^{H_2} f(x,y,z)\, dz = \int_{H_1}^{H_2} \frac{\partial}{\partial x} \{f(x,y,z)\}\, dz + f(x,y,H_2) \frac{\partial H_2}{\partial x} - f(x,y,H_1) \frac{\partial H_1}{\partial x} \tag{2.12}$$

to the equation (2.11) and simplifying, we have

$$\int_{H_1}^{H_2} \frac{\partial \rho}{\partial t}\, dz + \int_{H_1}^{H_2} \frac{\partial}{\partial x} (\rho u)\, dz + \int_{H_1}^{H_2} \frac{\partial}{\partial y} (\rho v)\, dz - (\rho u)_2 \frac{\partial H_2}{\partial x} - (\rho v)_2 \frac{\partial H_2}{\partial y} + (\rho u)_1 \frac{\partial H_1}{\partial x} + (\rho v)_1 \frac{\partial H_1}{\partial y} + [\rho w]_{H_1}^{H_2} = 0 \tag{2.13}$$

The integrals of (ρu) and (ρv) can be evaluated by integration by part to give the following equation :

$$\int_{H_1}^{H_2} \frac{\partial \rho}{\partial t}\, dz + H_2 \left[\frac{\partial}{\partial x} (\rho u)_2 + \frac{\partial}{\partial y} (\rho v)_2\right] - H_1 \left[\frac{\partial}{\partial x} (\rho u)_1 + \frac{\partial}{\partial y} (\rho v)_1\right] - \frac{\partial}{\partial x}\left[\int_{H_1}^{H_2} \left(\rho z \frac{\partial u}{\partial z} + uz \frac{\partial \rho}{\partial z}\right) dz\right] - \frac{\partial}{\partial y} \left[\int_{H_1}^{H_2} \left(\rho z \frac{\partial v}{\partial z} + vz \frac{\partial \rho}{\partial z}\right) dz\right] + [\rho w]_{H_1}^{H_2} = 0 \tag{2.14}$$

Introducing the expressions for u,v and their derivatives in the above equation (2.14), we get,

$$\frac{\partial}{\partial x}[(F_2 + G_1)\frac{\partial p}{\partial x}] + \frac{\partial}{\partial y}[(F_2' + G_1')\frac{\partial p}{\partial y}]$$

$$= H_2 [\frac{\partial}{\partial x}(\rho u)_2 + \frac{\partial}{\partial y}(\rho v)_2] - H_1[\frac{\partial}{\partial x}(\rho u)_1 + \frac{\partial}{\partial y}(\rho v)_1]$$

$$- \frac{\partial}{\partial x}[\frac{(U_2 - U_1)(F_3 + G_2)}{F_o} + U_1 G_3]$$

$$- \frac{\partial}{\partial y}[\frac{(V_2 - V_1)(F_3 + G_2')}{F_o'} + V_1 G_3]$$

$$+ \int_{H_1}^{H_2} \frac{\partial \rho}{\partial t} dz + [\rho w]_{H_1}^{H_2} \tag{2.15}$$

where

$$F_2 = \int_{H_1}^{H_2} \frac{\rho z}{\eta}(z - \frac{F_1}{F_o}) dz$$

$$F_2' = \int_{H_1}^{H_2} \frac{\rho z}{\eta}(z - \frac{F_1'}{F_o'}) dz$$

$$F_3 = \int_{H_1}^{H_2} \frac{\rho z}{\eta} dz$$

$$G_1 = \int_{H_1}^{H_2} [z \frac{\partial \rho}{\partial z}\{(\alpha_1 H_1 + \int_{H_1}^{z} \frac{z\, dz}{\eta}) - \frac{F_1}{F_o}(\alpha_1 + \int_{H_1}^{z} \frac{dz}{\eta})\}] dz$$

$$G_1' = \int_{H_1}^{H_2} [z \frac{\partial \rho}{\partial z}\{(\beta_1 H_1 + \int_{H_1}^{z} \frac{z\, dz}{\eta}) - \frac{F_1'}{F_o'}(\beta_1 + \int_{H_1}^{z} \frac{dz}{\eta})\}] dz$$

(2.16)

$$G_2 = \int_{H_1}^{H_2} [z \frac{\partial \rho}{\partial z}(\alpha_1 + \int_{H_1}^{z} \frac{dz}{\eta})] dz$$

$$G_2' = \int_{H_1}^{H_2} [z \frac{\partial \rho}{\partial z} (\beta_1 + \int_{H_1}^{z} \frac{dz}{\eta})] dz$$

$$G_3 = \int_{H_1}^{H_2} [z \frac{\partial \rho}{\partial z}] dz.$$

Equation (2.15) represents a generalized form of Reynolds equation for a fluid film-lubrication where slip-velocities are considered at the bearing surfaces. The two sets of functions F and G depend upon the variation of fluid properties along as well as across the film and slip conditions at the surfaces.

In the case of no-slip at the boundaries i.e.

$$\lambda_1 = \lambda_2 = \delta_1 = \delta_2 = 0$$

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 ,$$

the generalized equation (2.15) and equations (2.16) reduce to the following forms

$$\frac{\partial}{\partial x} [(F_2 + G_1) \frac{\partial p}{\partial x}] + \frac{\partial}{\partial y} [(F_2 + G_1) \frac{\partial p}{\partial y}]$$

$$= H_2 [\frac{\partial}{\partial x} (\rho u)_2 + \frac{\partial}{\partial y} (\rho v)_2] - H_1 [\frac{\partial}{\partial x} (\rho u)_1 + \frac{\partial}{\partial y} (\rho v)_1]$$

$$- \frac{\partial}{\partial x} [\frac{(U_2 - U_1)(F_3 + G_2)}{F_0} + U_1 G_3] - \frac{\partial}{\partial y} [\frac{(V_2 - V_1)(F_3 + G_2)}{F_0} + V_1 G_3]$$

$$+ \int_{H_1}^{H_2} \frac{\partial \rho}{\partial t} dz + [\rho w]_{H_1}^{H_2} \qquad (2.17)$$

where

$$F_o = \int_{H_1}^{H_2} \frac{dz}{\eta}$$

$$F_1 = \int_{H_1}^{H_2} \frac{z\,dz}{\eta}$$

$$F_2 = \int_{H_1}^{H_2} \frac{\rho z}{\eta} \left(z - \frac{F_1}{F_o}\right) dz \qquad (2.18)$$

$$F_3 = \int_{H_1}^{H_2} \frac{\rho z\,dz}{\eta}$$

$$G_1 = \int_{H_1}^{H_2} \left[z \frac{\partial\rho}{\partial z} \left(\int_{H_1}^{z} \frac{z\,dz}{\eta} - \frac{F_1}{F_o} \int_{H_1}^{z} \frac{dz}{\eta} \right) \right] dz$$

$$G_2 = \int_{H_1}^{H_2} \left[z \frac{\partial\rho}{\partial z} \left(\int_{H_1}^{z} \frac{dz}{\eta} \right) \right] dz$$

$$G_3 = \int_{H_1}^{H_2} \left[z \frac{\partial\rho}{\partial z} \right] dz$$

From above equation (2.17), a generalized form of Reynolds equation derived by Dowson [1962] can be deduced by considering $H_1 = 0$, and $H_2 = h$. The various forms of equations of fluid-film lubrication derived by Reynolds [1886] , Cope [1949] , and Zienkiewicz [1957] can also be deduced from the generalized equation (2.15).

In the following some particular cases are discussed.

(I) GAS LUBRICATION : HYDROSTATIC BEARING

(II) MULTIPLE VISCOUS INCOMPRESSIBLE LAYERS LUBRICATION : SLIDER BEARING

(III) THREE LAYERS LUBRICATION : HYDROSTATIC BEARING

(IV) SINGLE VISCOUS INCOMPRESSIBLE LAYER LUBRICATION WITH POROSITY.

2.3 GAS LUBRICATION : HYDROSTATIC BEARING

In the case of a single layer compressible lubrication and considering the conditions

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial z} = \frac{\partial \eta}{\partial z} = 0 \qquad \text{i.e.} \quad \rho = \rho(x,y), \quad \eta = \eta(x,y)$$

$$\lambda_1 = \delta_1 , \ \lambda_2 = \delta_2 , \tag{2.19}$$

the generalized equation with slip (2.15) reduces to the following form

$$\frac{\partial}{\partial x}\left[F_2 \frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[F_2 \frac{\partial p}{\partial y}\right]$$

$$= H_2 \left[\frac{\partial}{\partial x}(\rho u)_2 + \frac{\partial}{\partial y}(\rho v)_2\right] - H_1 \left[\frac{\partial}{\partial x}(\rho u)_1 + \frac{\partial}{\partial y}(\rho v)_1\right] \tag{2.20}$$

where

$$F_0 = \alpha_1 + \alpha_2 + \frac{H_2 - H_1}{\eta}$$

$$F_1 = \alpha_1 H_1 + \alpha_2 H_2 + \frac{(H_2^2 - H_1^2)}{2\eta}$$

$$F_2 = \rho\left[\left(\frac{H_2^3 - H_1^3}{3\eta}\right) - \frac{F_1}{F_0}\left(\frac{H_2^2 - H_1^2}{2\eta}\right)\right]$$

$$F_3 = \rho\left(\frac{H_2^2 - H_1^2}{2\eta}\right) \tag{2.21}$$

$$(\rho u)_1 = \rho\alpha_1\left(H_1 - \frac{F_1}{F_0}\right)\frac{\partial p}{\partial x} + \rho\left[U_1 + \frac{\alpha_1 (U_2 - U_1)}{F_0}\right]$$

$$(\rho u)_2 = -\rho\alpha_2(H_2 - \frac{F_1}{F_o})\frac{\partial p}{\partial x} + \rho[U_2 - \alpha_2\frac{(U_2 - U_1)}{F_o}]$$

$$(\rho v)_1 = \rho\alpha_1(H_1 - \frac{F_1}{F_o})\frac{\partial p}{\partial y} + \rho[V_1 + \alpha_1\frac{(V_2 - V_1)}{F_o}]$$

$$(\rho v)_2 = -\rho\alpha_2(H_2 - \frac{F_1}{F_o})\frac{\partial p}{\partial y} + \rho[V_2 - \alpha_2\frac{(V_2 - V_1)}{F_o}]$$

By using equations (2.21) and remembering that in the case of two-dimensional slider bearing,

$$[\rho w]_{H_1}^{H_2} = (\rho u)_2\frac{\partial H_2}{\partial x} + (\rho v)_2\frac{\partial H_2}{\partial y} - (\rho u)_1\frac{\partial H_1}{\partial x} - (\rho v)_1\frac{\partial H_1}{\partial y} - \rho V \tag{2.22}$$

where V-squeeze velocity in the (-ve) z direction, the equation (2.20) can be simplified as follows :

$$\frac{\partial}{\partial x}[\{\frac{\rho}{\eta}(\frac{H_2^3 - H_1^3}{3} + \lambda_2 H_2^2 + \lambda_1 H_1^2) - \frac{\rho F_1^2}{F_o}\ \frac{\partial p}{\partial x}]$$

$$+\frac{\partial}{\partial y}[\{\frac{\rho}{\eta}(\frac{H_2^3 - H_1^3}{3}, + \lambda_2 H_2^2 + \lambda_1 H_1^2) - \frac{\rho F_1^2}{F_o}\ \frac{\partial p}{\partial y}]$$

$$= \frac{\partial}{\partial x}[\rho U_2(H_2 - \frac{F_1}{F_o}) - \rho U_1(H_1 - \frac{F_1}{F_o})]$$

$$+\frac{\partial}{\partial y}[\rho V_2(H_2 - \frac{F_1}{F_o}) - \rho V_1(H_1 - \frac{F_1}{F_o})]$$

$$- \rho V \tag{2.23}$$

Equation (2.23) can be further simplified again by considering

$$H_1 = 0 \ , \ H_2 = h$$

$$\lambda_1 = \lambda_2 = \lambda \quad \text{i.e.} \quad \alpha_1 = \alpha_2 = \frac{\lambda}{\eta} \tag{2.24}$$

$$U_1 = U \;,\; U_2 = V_1 = V_2 = 0$$

as follows :

$$\frac{\partial}{\partial x}\left[\frac{\rho h^3}{12\eta}\left(1 + 6\,\frac{\lambda}{h}\right)\frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\rho h^3}{12\eta}\left(1 + 6\,\frac{\lambda}{h}\right)\frac{\partial p}{\partial y}\right]$$

$$= \frac{U}{2}\frac{\partial}{\partial x}(\rho h) - \rho V \tag{2.25}$$

This equation is same as that obtained by Burgdorfer [1959] when V = 0.

Assuming steady state and isothermal condition in the gas film, we can write,

$$\frac{\rho}{\rho_a} = \frac{p}{p_a} \tag{2.26}$$

Further, from equation (2.2) we can also have,

$$\frac{\lambda}{\lambda_a} = \frac{p_a}{\eta_a}\frac{\eta}{p} \tag{2.27}$$

where suffix 'a' denotes the respective reference quantities.

Using equations (2.26) and (2.27), the equation (2.25) can finally be written as follows :

$$\frac{\partial}{\partial x}\left[\left\{\frac{ph^3}{12\eta}\left(1 + 6\,\frac{\lambda_a\, p_a}{\eta_a\, h}\,\frac{\eta}{p}\right)\frac{\partial p}{\partial x}\right\}\right]$$

$$+ \frac{\partial}{\partial y}\left[\left\{\frac{ph^3}{12\eta}\left(1 + 6\,\frac{\lambda_a\, p_a}{\eta_a\, h}\,\frac{\eta}{p}\right)\frac{\partial p}{\partial y}\right\}\right] = \frac{U}{2}\frac{\partial}{\partial x}(ph) - pV \tag{2.28}$$

This is the generalized form of Reynolds equation with slip and viscosity variation.

Now, let us consider the flow of a gas, with constant viscosity, in the case of hydrostatic bearing as shown in the Fig. (2.2). The Reynolds equation for this case can be written, from equation (2.28), as follows :

$$\frac{\partial}{\partial r}\left[\frac{rh^3}{12\eta}\left(p + \frac{6\,\lambda_a\,p_a}{h}\right)\frac{\partial p}{\partial r}\right] = 0 \tag{2.29}$$

where p_a, λ_a may be taken as the reference values in the recess of the bearing $0 \leq r \leq r_i$.

The flow flux can be written as

$$Q = -\frac{\pi r\,h^3}{6\,\eta}\left[1 + \frac{6}{h}\frac{\lambda_a\,p_a}{p}\frac{\partial p}{\partial r}\right] \tag{2.30}$$

which is a function of r. Combining equations (2.29) and (2.30), it can be noted that

$$\frac{\partial}{\partial r}\left[-\frac{\bar{p}}{2\pi}Q\right] = 0 \tag{2.31}$$

which implies that in the case of compressible lubrication,

$$p\,Q = p_a\,Q_a = \text{constant} \tag{2.32}$$

where Q_a is the inlet flux in the recess $0 \leq r \leq r_i$.

Now from equations (2.30) and (2.32), we have

$$\frac{\partial}{\partial r}\left[p + \frac{6\lambda_a\,p_a}{h}\right]^2 = -\frac{12\eta\,p_a\,Q_a}{\pi h^3}\frac{1}{r} \tag{2.33}$$

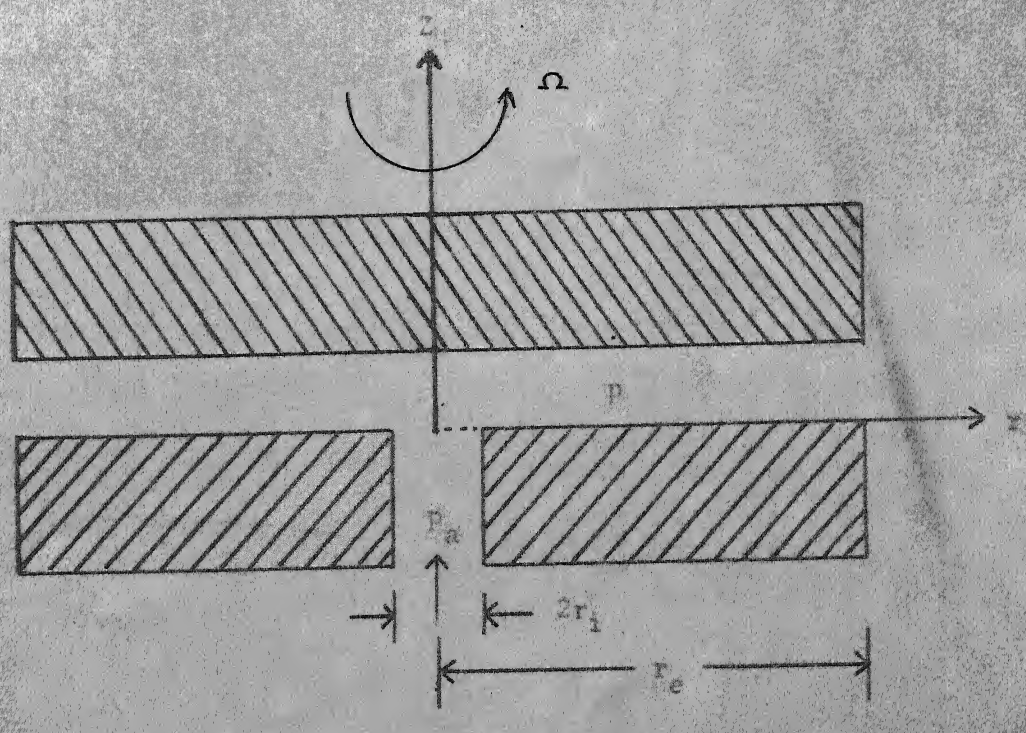

Fig. 2.2. HYDROSTATIC BEARING WITH GAS LUBRICATION

The boundary condition for p are

$$p = p_a \text{ at } r = r_i$$

$$p = 0 \text{ at } r = r_e$$

Integrating equation (2.33) and using the above boundary condition the expression for inlet flux and the film pressure can be obtained as follows :

$$Q_a = \frac{\pi h^3 p_a}{12\eta \ln \frac{r_e}{r_i}} \left(1 + \frac{12\lambda_a}{h}\right) \tag{2.34}$$

and

$$\frac{p}{p_a} = \left[\left(1 + \frac{6\lambda_a}{h}\right)^2 \frac{\ln \frac{r_e}{r}}{\ln \frac{r_e}{r_i}} + \left(\frac{6\lambda_a}{h}\right)^2 \frac{\ln \frac{r}{r_i}}{\ln \frac{r_e}{r_i}}\right]^{\frac{1}{2}} - \frac{6\lambda_a}{h} \tag{2.35}$$

It can be seen from equation (2.35) that as λ_a increases the inlet flux Q_a increases for a given inlet pressure p_a.

Now from equation (2.35), we have

$$\frac{d}{d\lambda_a}\left(\frac{p}{p_a}\right) = -\frac{6}{h}\left[1 - \frac{\frac{\ln \frac{r_e}{r}}{\ln \frac{r_e}{r_i}} + \frac{6\lambda_a}{h}}{\left\{\left(1 + 12\frac{\lambda_a}{h}\right)\frac{\ln \frac{r_e}{r}}{\ln \frac{r_e}{r_i}} + \left(\frac{6\lambda_a}{h}\right)^2\right\}^{\frac{1}{2}}}\right] \tag{2.36}$$

Since, $\frac{\ln \frac{r_e}{r}}{\ln \frac{r_e}{r_i}} < 1$, it can be noted that the expression inside the square bracket of the equation (2.36) is positive. Hence $\frac{d}{d\lambda_a}\left(\frac{p}{p_a}\right)$

is negative. Thus, it can be concluded that the pressure decreases as λ_a, the mean free path increases. Further, as the load capacity is the integral of the pressure, it also decreases as λ_a increases.

2.4 MULTIPLE VISCOUS INCOMPRESSIBLE LAYERS LUBRICATION : SLIDER BEARING

It has already been pointed out that the viscosity varies across the film-thickness due to thermal changes and such a study has been conducted by Qvale and Wiltshire [1972] . The viscosity of the lubricant also changes with concentration of the additive which may vary across the film-thickness, Shukla [1972] and Isa [1974] . Further the viscosity of the lubricant near the solid surface may increase considerably the closer the boundary is approached as pointed out by Devenport [1973, p 19].

Keeping this in view, a new concept of multiple layers lubrication is proposed here so that the effects of viscosity variation across the film and near the solid surface can be taken into account, Qvale and Wiltshire [1972], Askwith et. al. [1966] . The Reynolds equation presented here can also be used to study the effects of surfactant at the bearing surface (such as fatty acids) which might change the viscosity near the surface, Askwith et. al. [1966] and Fein and Kreuz [1966] . To simplify the matter, the study here is confined to only multiple incompressible layers, however there is no difficulty in deriving a corresponding Reynolds equation for multiple compressible layers.

In the case of incompressible lubricant, all G-functions vanish and generalized equation with slip (2.15) simplies to the following form

$$\frac{\partial}{\partial x}\left[F_2 \frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[F_2' \frac{\partial p}{\partial y}\right]$$

$$= H_2 \left[\frac{\partial}{\partial x}(u)_2 + \frac{\partial}{\partial y}(v)_2\right] - H_1\left[\frac{\partial}{\partial x}(u)_1 + \frac{\partial}{\partial y}(v)_1\right]$$

$$- \frac{\partial}{\partial x}\left[\frac{(U_2 - U_1) F_3}{F_o}\right] - \frac{\partial}{\partial y}\left[\frac{(V_2 - V_1) F_3}{F_o'}\right] + [w]_{H_1}^{H_2} \qquad (2.37)$$

where

$$F_o = \alpha_1 + \alpha_2 + \int_{H_1}^{H_2} \frac{dz}{\eta}$$

$$F_o' = \beta_1 + \beta_2 + \int_{H_1}^{H_2} \frac{dz}{\eta}$$

$$F_1 = \alpha_1 H_1 + \alpha_2 H_2 + \int_{H_1}^{H_2} \frac{z\,dz}{\eta}$$

$$F_1' = \beta_1 H_1 + \beta_2 H_2 + \int_{H_1}^{H_2} \frac{z\,dz}{\eta} \qquad (2.38)$$

$$F_2 = \int_{H_1}^{H_2} \frac{z}{\eta}\left(z - \frac{F_1}{F_o}\right) dz$$

$$F_2' = \int_{H_1}^{H_2} \frac{z}{\eta}\left(z - \frac{F_1'}{F_o'}\right) dz$$

$$F_3 = \int_{H_1}^{H_2} \frac{z\,dz}{\eta}$$

To study a particular case of the equation (2.38), let us consider the flow of three fluid layers in the case of a slider

bearing whose configuration is shown in Fig. (2.3). Considering also the effects of slip at the surfaces in this case and remembering equations (2.1) and (2.3), we have

$$\lambda = \frac{\eta_1}{\beta}$$

where η_1 is the viscosity of the lubricant at the surface and β is the coefficient of sliding friction at the surface. It is noted here, that as β increases λ decreases and β tends to infinity in the case of no-slip at the surface. Now, keeping in view the physical situation represented in Fig. (2.3) and considering the following

$$\begin{aligned} &U_1 = U,\ U_2 = V_1 = V_2 = 0 \\ &\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \frac{1}{\beta} \\ &\rho = \text{const},\ \eta_1 = \eta_2 = \text{const} \end{aligned} \qquad (2.39)$$

the Reynolds equation applicable to this situation can be derived from equation (2.37), as follows :

$$\begin{aligned} \frac{\partial}{\partial x}\left[F_2 \frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[F_2 \frac{\partial p}{\partial y}\right] \\ = (h+H)\left[\frac{\partial}{\partial x}(u)_2 + \frac{\partial}{\partial y}(v)_2\right] \\ + U\frac{\partial}{\partial x}\left(\frac{F_3}{F_0}\right) + [w]_0^{h+H} \end{aligned} \qquad (2.40)$$

where

$$F_0 = \frac{h}{\eta_2} + \frac{H}{\eta_1} + \frac{2}{\beta}$$

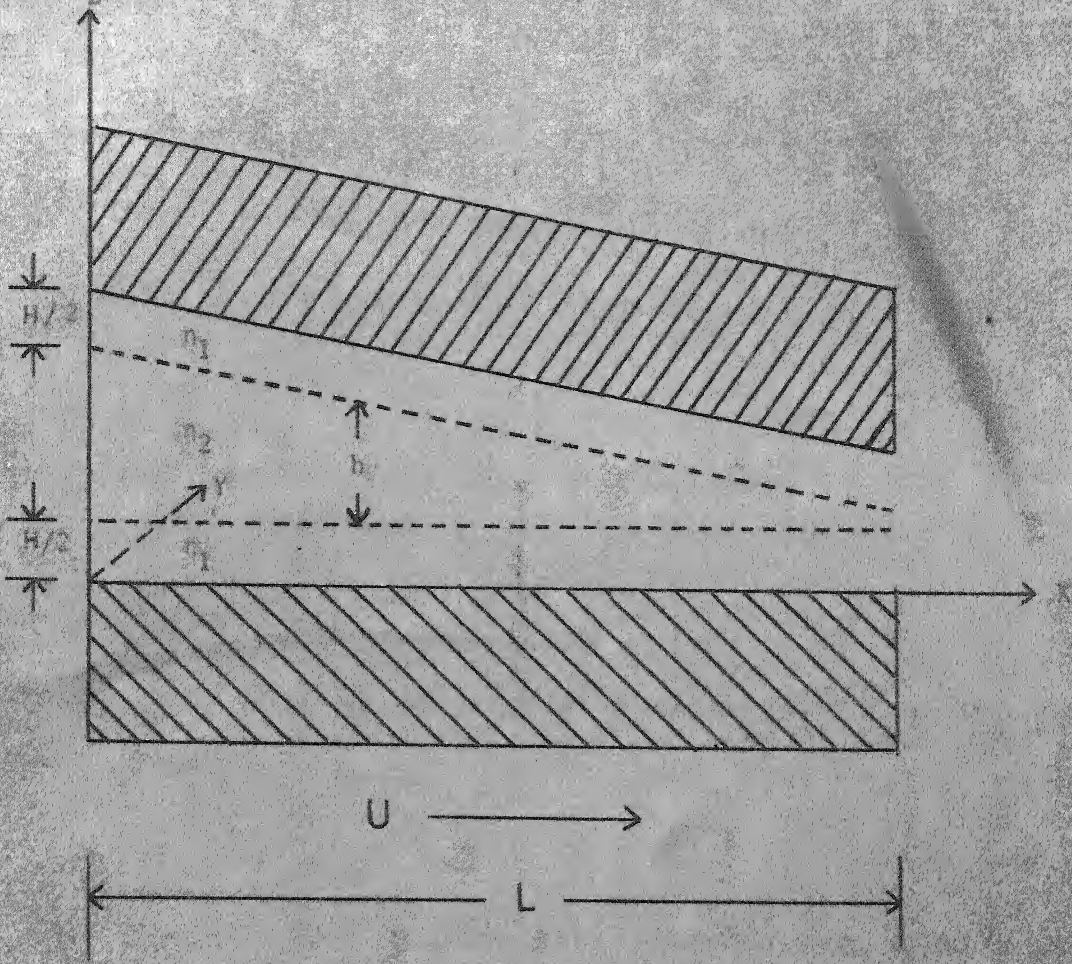

Fig. 2.3 SLIDER BEARING WITH THREE LAYERS OF LUBRICANT

$$F_2 = \frac{h^3}{12\eta_2} + \frac{H^3 + 3H^2h + 3Hh^2}{12\eta_1}$$

$$F_3 = \left(\frac{h+H}{2}\right)\left(\frac{h}{\eta_2} + \frac{H}{\eta_1}\right)$$

$$(u)_2 = -\frac{1}{\beta}\left(\frac{h+H}{2}\right)\frac{\partial p}{\partial x} + \frac{U}{\beta F_o} \qquad (2.41)$$

$$(v)_2 = -\frac{1}{\beta}\left(\frac{h+H}{2}\right)\frac{\partial p}{\partial y}$$

$$[w]_o^{h+H} = (u)_2 \frac{\partial}{\partial x}(h+H) + (v)_2 \frac{\partial}{\partial y}(h+H) - V$$

Simplifying further equation (2.40) by using equation (2.41), we get,

$$\frac{\partial}{\partial x}\left[F_4 \frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[F_4 \frac{\partial p}{\partial y}\right] = \frac{U}{2}\frac{\partial}{\partial x}(h+H) - V \qquad (2.42)$$

where

$$F_4 = \frac{h^3}{12\eta_2} + \frac{H^3+3H^2h+3Hh^2}{12\eta_1} + \frac{(h+H)^2}{2\beta} \qquad (2.43)$$

Applying equation (2.42) to the case of one dimensional bearing, we have

$$\frac{d}{dx}\left[F_4 \frac{dp}{dx}\right] = \frac{U}{2}\frac{d}{dx}(h+H) - V \qquad (2.44)$$

Integrating above equation (2.44) with the boundary condition (keeping H constant)

$$\frac{dp}{dx} = 0 \quad \text{at } x = x_o \;, \; h = h_o \qquad (2.45)$$

we get,

$$\frac{dp}{dx} = \frac{\frac{U}{2}(h-h_o) + V(x_o-x)}{F_4} \qquad (2.46)$$

By following the usual procedure, the load capacity and the force of friction in this case can be written as follows :

$$W = \int_0^L \left(-x \frac{dp}{dx}\right) dx = -\int_0^L x \frac{\frac{U}{2}(h-h_o) + V(x_o-x)}{F_4} dx \qquad (2.47)$$

$$F = \int_0^L \eta_1 \left(\frac{\partial u}{\partial z}\right)_{z=0} dx$$

$$= \int_0^L \left[\frac{U}{F_o} + \left(\frac{h+H}{2}\right) \frac{\frac{U}{2}(h-h_o) + V(x_o-x)}{F_4} \right] dx \qquad (2.48)$$

From the expressions of F_o and F_4 it can be seen that F_o and F_4 decrease as β , η_1 or η_2 increase. Hence from equations (2.47) and (2.48) it can be concluded that the load carrying capacity and the frictional force increase as β, η_1 or η_2 increase.

In the case of pure squeezing between two parallel plate, equation (2.44) can be integrated with usual boundary conditions to give the expression for film pressure and load capacity as follows :

$$p = \frac{V}{2F_4} x(L-x) \qquad (2.49)$$

$$W = \int_0^L p \, dx = \frac{VL^3}{12 F_4} \qquad (2.50)$$

As in the previous case it can also be seen that load carrying capacity increases as β, η_1 , η_2 increase. Hence, it is concluded that the load capacity and friction force increase as the viscosities of the layers as well as the coefficient of the sliding

friction increase. The effect of slip is to decrease the load capacity and the force of friction.

2.5 THREE LAYERS LUBRICATION : HYDROSTATIC BEARING

Consider the flow of three incompressible fluid layers in the case of externally pressurized hydrostatic bearing as shown in Fig. (2.4). The equation governing the flow in this case can be calculated as follows :

$$\frac{d}{dr}\left[r\, F_4\, \frac{dp}{dr}\right] = 0 \tag{2.51}$$

where F_4 is defined in the equation (2.43).

Integrating equation (2.51) with respect to r and using the boundary conditions

$$\begin{aligned} p &= p_i \quad \text{at } r = r_i \\ p &= 0 \quad \text{at } r = r_e \end{aligned} \tag{2.52}$$

we have,

$$\begin{aligned} p &= p_i - \frac{Q}{2\pi F_4}\, \ln \frac{r}{r_i} \\ p_i &= \frac{Q}{2\pi F_4}\, \ln \frac{r_e}{r_i} \,. \end{aligned} \tag{2.53}$$

The load capacity can be evaluated from the expression

$$W = \pi p_i\, r_i^2 + \int_{r_i}^{r_e} 2\pi\, r\, p\, dr \tag{2.54}$$

as follows :

$$W = \frac{Q(r_e^2 - r_i^2)}{4\pi\, F_4} \tag{2.55-a}$$

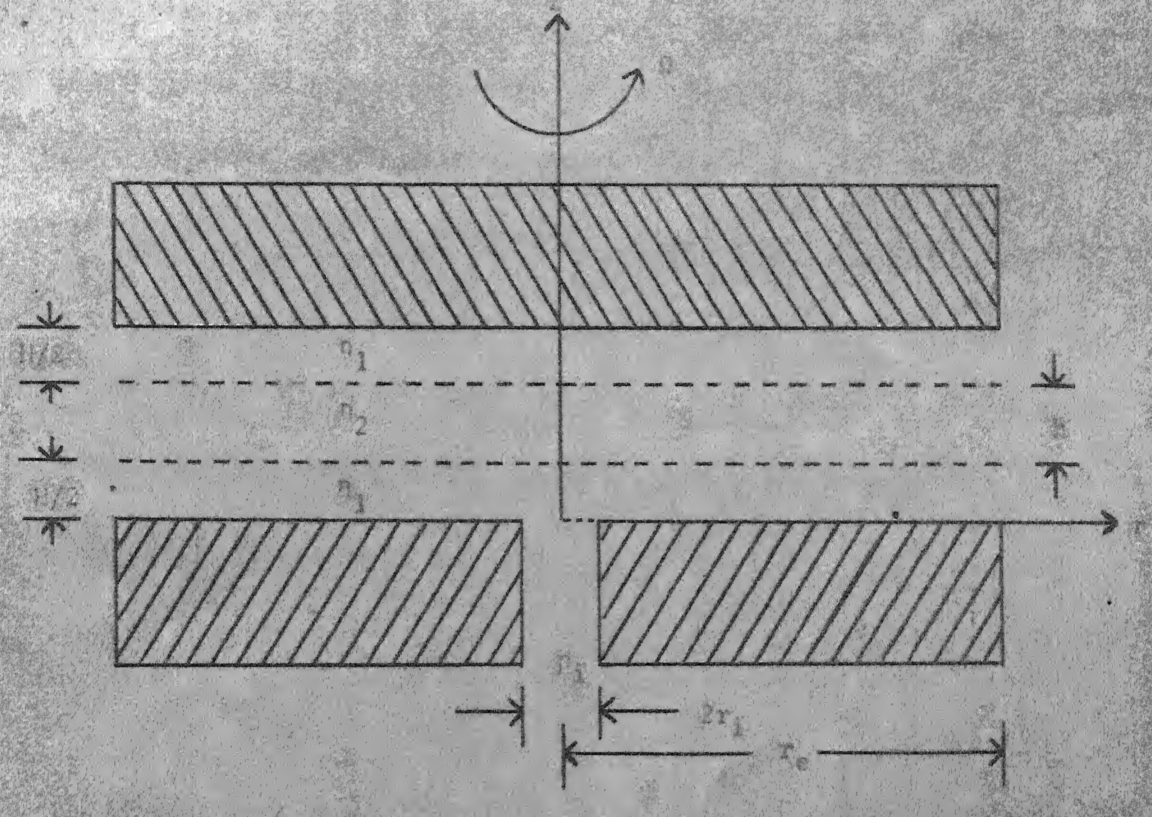

FIG. 2.4 HYDROSTATIC BEARING WITH THREE LAYERS OF LUBRICANT

or

$$W = \frac{p_i(r_e^2 - r_i^2)}{2\ \ell n\ (\frac{r_e}{r_i})} \qquad (2.55\text{-b})$$

It has already been noted from the equation (2.43) that F_4 decreases as β, the coefficient sliding friction increases. Hence, it can be seen from the equation (2.55-a) that for a fixed flux Q in a hydrostatic bearing, the load capacity increases as viscosities of the layers increase and it decreases with slip at the surface.

2.6 SINGLE VISCOUS INCOMPRESSIBLE LAYER LUBRICATION WITH POROSITY

Let us now consider the case of single incompressible **layer** lubrication with slip velocity at the porous boundary. Taking,

$$\begin{aligned} & H_1 = 0\ ,\ H_2 = h,\ V_1 = V_2 = 0,\ \eta = \text{const.},\ \rho = \text{const.} \\ & \lambda_1 = \delta_1 = 0,\ \lambda_2 = \delta_2 = \lambda \\ & \text{i.e. } \alpha_1 = \beta_1 = 0,\ \alpha_2 = \beta_2 = \frac{\lambda}{\eta} \\ & [w]_{H_1}^{H_2} = [w]_0^h = (u)_2 \frac{\partial h}{\partial x} + (v_2) \frac{\partial h}{\partial y} - V_s \end{aligned} \qquad (2.56)$$

the Reynolds equation (2.37), with slip in the two dimensional form can be written as follows :

$$\frac{\partial}{\partial x}[F_2 \frac{\partial p}{\partial x}] + \frac{\partial}{\partial y}[F_2 \frac{\partial p}{\partial y}] = \frac{\partial}{\partial x}[h(u)_2] + \frac{\partial}{\partial y}[h(v)_2] - \frac{\partial}{\partial x}[\frac{(U_2 - U_1)F_3}{F_0}] - V_s \qquad (2.57)$$

where

$$(u)_2 = -\frac{\lambda}{\eta}\,(h - \frac{F_1}{F_o})\;\frac{\partial p}{\partial x} + [U_2 - \frac{\lambda}{\eta}\,(\frac{U_2 - U_1}{F_o})]$$

$$(v)_2 = -\frac{\lambda}{\eta}\,(h - \frac{F_1}{F_o})\,\frac{\partial p}{\partial y}$$

$$F_o = \frac{h+\lambda}{\eta}\;,\; F_1 = \frac{h(h+2\lambda)}{2\eta}\;, \qquad (2.58)$$

$$F_2 = \frac{h^3}{12\eta}\,(\frac{h - 2\lambda}{h + \lambda})\;,\; F_3 = \frac{h^2}{2\eta}$$

$$V_s = V + \frac{\phi}{\eta}\,(\frac{\partial p^*}{\partial z})_{z=h}\;,$$

and p^* is the pressure in the porous region.

Simplifying equation (2.57) by using expressions (2.58) we have

$$\frac{\partial}{\partial x}\,[\frac{h^3}{\sigma_1}\,\frac{\partial p}{\partial x}] + \frac{\partial}{\partial y}\,[\frac{h^3}{\sigma_1}\,\frac{\partial p}{\partial y}]$$

$$= 6\eta\,\frac{\partial}{\partial x}\,[(\frac{h^2}{h+\lambda})\,U_2 + \frac{h(h+2\lambda)}{(h+\lambda)}\,U_1]$$

$$- 12\eta\,[V + \frac{\phi}{\eta}\,(\frac{\partial p^*}{\partial z})_{z=h}] \qquad (2.59)$$

where

$$\frac{1}{\sigma_1} = [1 + \frac{3}{1 + \frac{h}{\lambda}}] \qquad (2.60)$$

In this case, on comparing equations (2.1) and (2.4), $\lambda = \frac{[\phi]^{\frac{1}{2}}}{\xi}$, and the equation (2.60) can be rewritten as

$$\frac{1}{\sigma_1} = \{1 + \frac{3}{[1 + \frac{h\xi}{(\phi)^{\frac{1}{2}}}]}\} \qquad (2.60\text{-a})$$

It is noted here that when the slip coefficient $\xi \to 0$, $\sigma_1 \to 0$ while when $\xi \to \infty$, $\sigma_1 \to 1$ (No Slip).

Now, it can be noted that the equation (2.59) reduces to the case of two dimensional squeeze film between porous and a solid rectangular plates as discussed by Wu [1972] .

In the case of externally pressurized porous squeeze film bearing, Wang [1975] ,

$$\frac{\phi}{\eta}\left(\frac{\partial p^*}{\partial z}\right)_{z=h} = \frac{\phi}{\eta}\left(\frac{p_e - p}{H}\right)$$

where H is the thickness of porous facing and p_e is the externally applied pressure, the Reynolds equation (2.59) can be modified to,

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\frac{12\eta\sigma_1}{h^3}\left[V + \frac{\phi}{\eta}\left(\frac{p_e - p}{H}\right)\right] \tag{2.61}$$

In the following, we study the case of one-dimensional externally pressurized porous squeeze film as shown in Fig. (2.5).

In this case, the equation governing the pressure can be written as

$$\frac{d^2 p}{dx^2} = -\frac{12\eta\sigma_1}{h^3}\left[V + \frac{\phi}{\eta}\left(\frac{p_e - p}{H}\right)\right] \tag{2.62}$$

which simplies to

$$\frac{d^2 p}{dx} - K^2 p = -K^2 p_e - S \tag{2.63}$$

where

$$K^2 = \frac{12\sigma_1\phi}{Hh^3}, \quad S = \frac{12\eta\sigma_1}{h^3} V$$

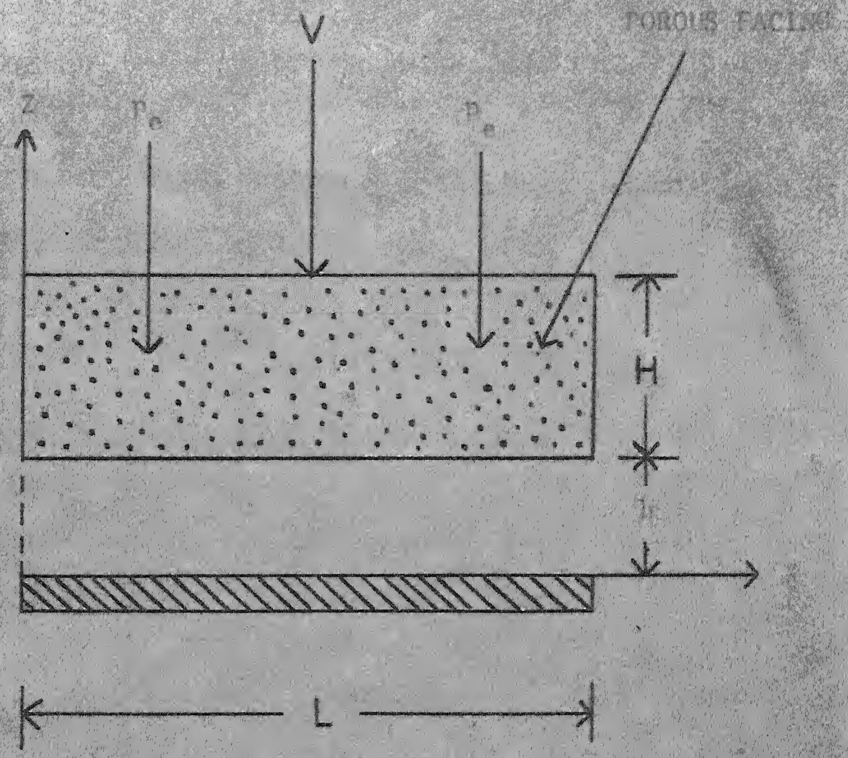

Fig. 2.5 EXTERNALLY PRESSURIZED POROUS SQUEEZE BEARING.

Solving equation (2.63) by using boundary conditions

$$p = 0 \quad \text{at } x = 0$$
$$\frac{\partial p}{\partial x} = 0 \quad \text{at } x = \frac{L}{2} \qquad (2.64)$$

we have,

$$p = [p_e + \frac{S}{K^2}]\,[1 - \frac{\cosh K(x - \frac{L}{2})}{\cosh \frac{KL}{2}}] \qquad (2.65)$$

The load capacity of the bearing is found by integrating the pressure in the film as follows :

$$W = \int_0^L p\,dx = L\,[p_e + \frac{S}{K^2}]\,[1 - \frac{\tanh \frac{KL}{2}}{\frac{KL}{2}}] \qquad (2.66)$$

In dimensionless form it yields

$$\bar{W} = [1 + \frac{\bar{V}\,.\,\bar{H}}{\bar{\phi}}]\,[1 - \frac{\tanh \nu}{\nu}] \qquad (2.67)$$

where

$$\bar{h} = \frac{h}{h_o} \;,\; \bar{W} = \frac{W}{L\,p_e} \;,\; \bar{\phi} = \frac{\phi L}{h_o^3}$$
$$\bar{H} = \frac{H}{L} \;,\; \bar{V} = -\frac{d\bar{h}}{d\bar{t}} \;,\; \bar{t} = \frac{p_e\,h_o^2}{\eta L^2}\,t \qquad (2.68)$$
$$\nu = (\frac{3\,\sigma_1\,\bar{\phi}}{\bar{H}\,\bar{h}^3})^{\frac{1}{2}} .$$

The time of squeezing for a constant load can be found as follows :

$$\bar{T} = \frac{\bar{H}}{\bar{\phi}} \int_{\bar{h}_1}^{1} \frac{d\bar{h}}{[\frac{\bar{W}}{(1 - \frac{\tanh \nu}{\nu})} - 1]} \qquad (2.69)$$

where $\bar{h}_1 = \frac{h_1}{h_o}$.

When $\sigma_1 = 1$ ($\xi=\infty$), the expressions for $\bar{W}$ and $\bar{T}$ reduce to the case corresponding to no-slip condition. The load capacity and the time of squeezing given by equations (2.67) and (2.69) are plotted in Figs. (2.6) to (2.11) for various parameters. Here the load capacity and the time of squeezing are calculated for $\xi= 0.1$, $\xi= 0.5$ and $\xi= \infty$.

It is clear from Fig. (2.6) that for no squeezing ($\bar{V} = 0$) the load capacity $\bar{W}$ decreases as $\frac{H}{L}$ increases and increases as ξ(the slip coefficient) increases for a fixed $\bar{\phi}$. But in the case of squeezing ($\bar{V} = 3.0$), [see Fig.(2.7)] , $\bar{W}$ decreases for a certain small values of $\frac{H}{L}$ and thereafter increases as $\frac{H}{L}$ increases. Further it is also noted from Fig. (2.7) that for small values of $\frac{H}{L}$ the load capacity is lower for small values of $\bar{\phi}$ while this situation is reversed for higher values of $\frac{H}{L}$. From Fig. (2.8) it can be seen that for $\frac{H}{L} = 0.05$, the load capacity increases as $\bar{\phi}$ increases for $\bar{V} = 0$ but it decreases for $\bar{V} = 3.0$.

It is observed from Fig. (2.9), (2.10) and (2.11) that the existance of slip at the surface shortens the time of squeezing between two plates. For a fixed $\frac{H}{L} = 0.1$, it can be seen from Fig. (2.9) that the time of squeezing increases as $\frac{h_1}{h_o}$ and $\bar{\phi}$ decrease . Further, from Fig. (2.9), it is noted that the time of squeezing increases as ξ increases. In Fig. (2.10), it is shown that the time of squeezing increases as $\frac{H}{L}$ increases for a fixed $\frac{h_1}{h_o} = 0.25$. From Fig. (2.11) for a fixed $\frac{h_1}{h_o} = 0.25$, it can be seen that the time of squeezing increases as $\bar{\phi}$ decreases but the value corresponds to $\frac{H}{L} = 0.1$ is still higher than that of $\frac{H}{L} = 0.01$.

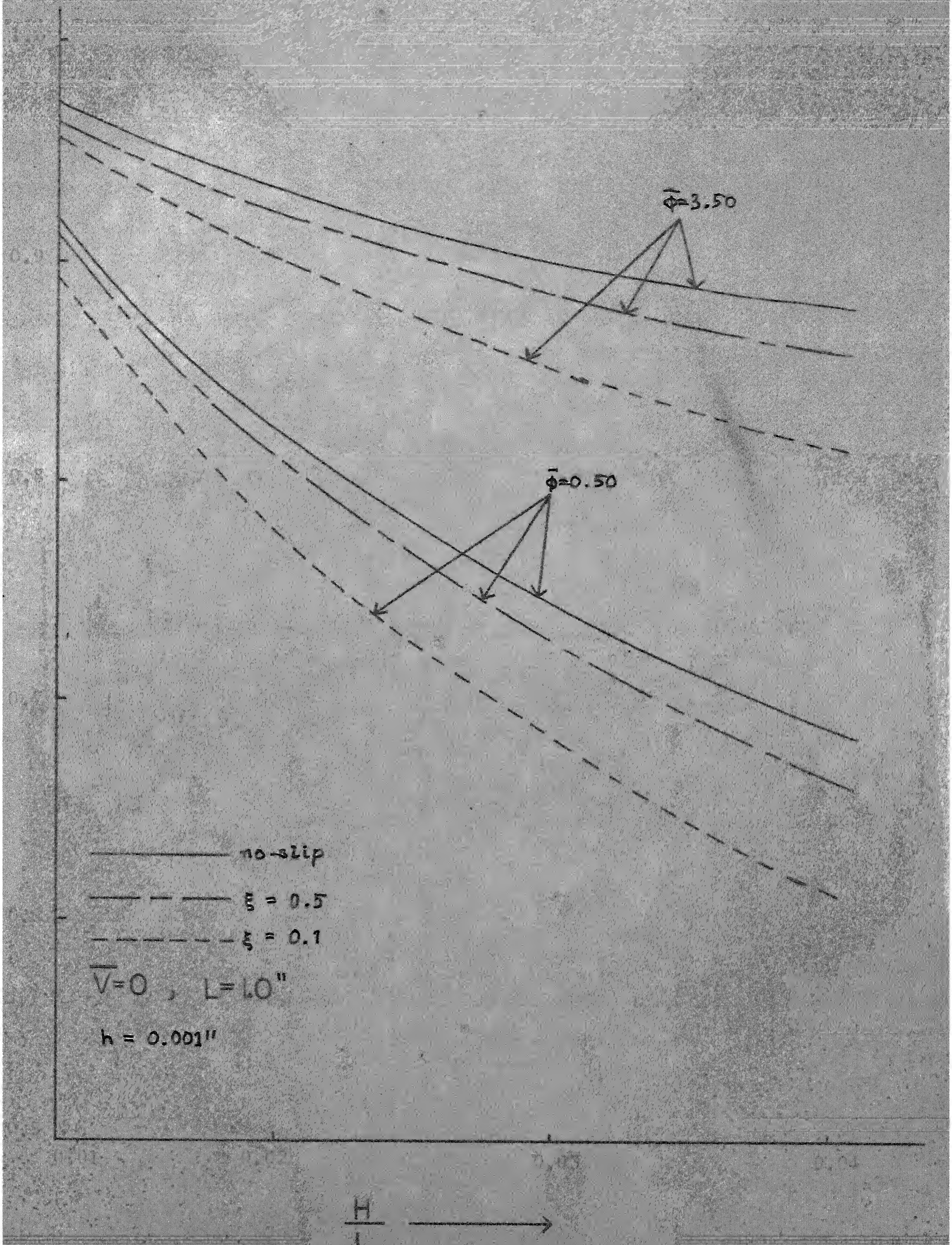

Fig. 2.6 FLUID FILM LOAD-CAPACITY

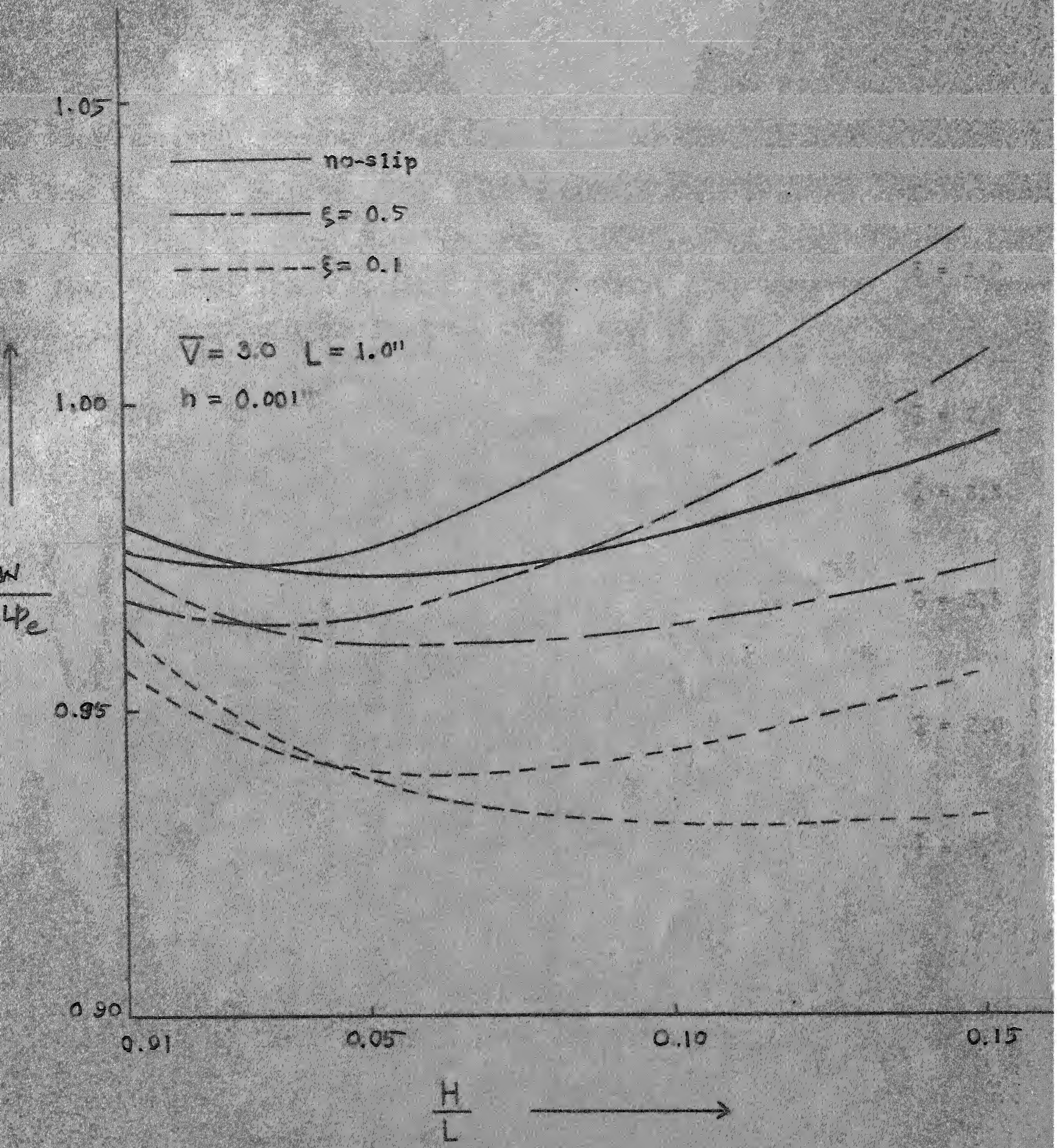

FIG. 2.7 SQUEEZE FILM LOAD CAPACITY

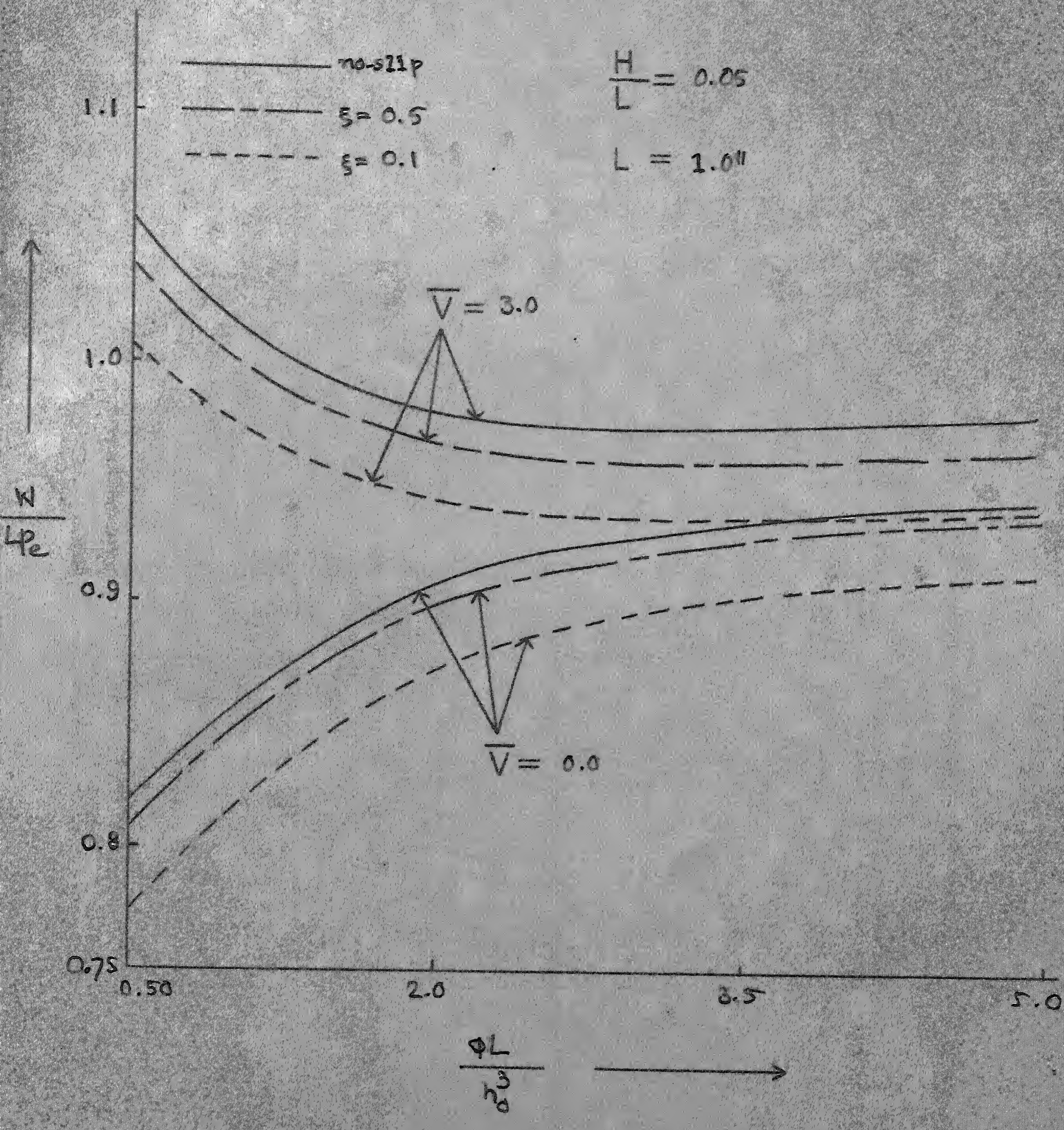

Fig. 2.8 LOAD CAPACITY AND PERMEABILITY PARAMETER RELATION

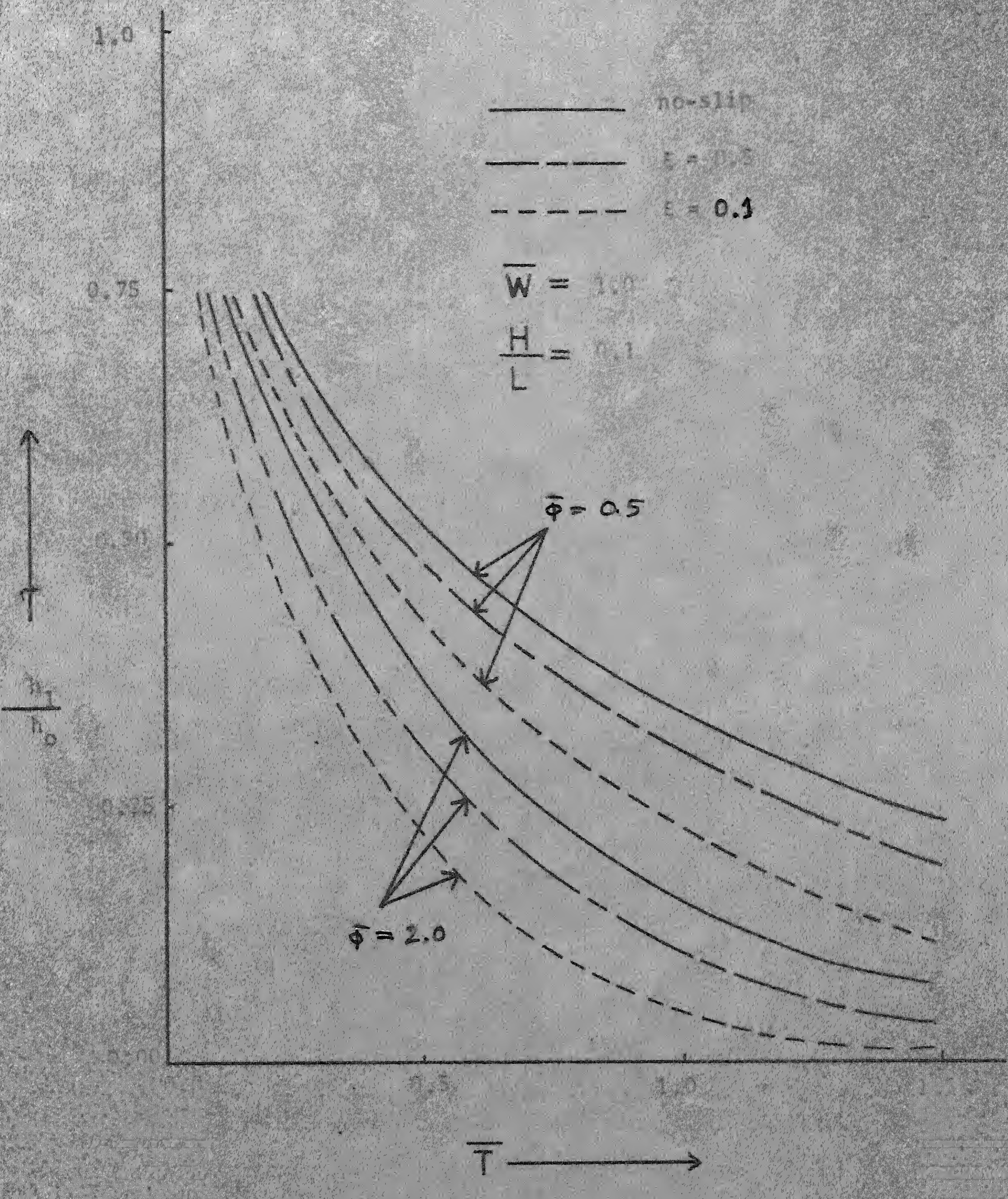

Fig. [illegible] FILM THICKNESS AND TIME RELATION

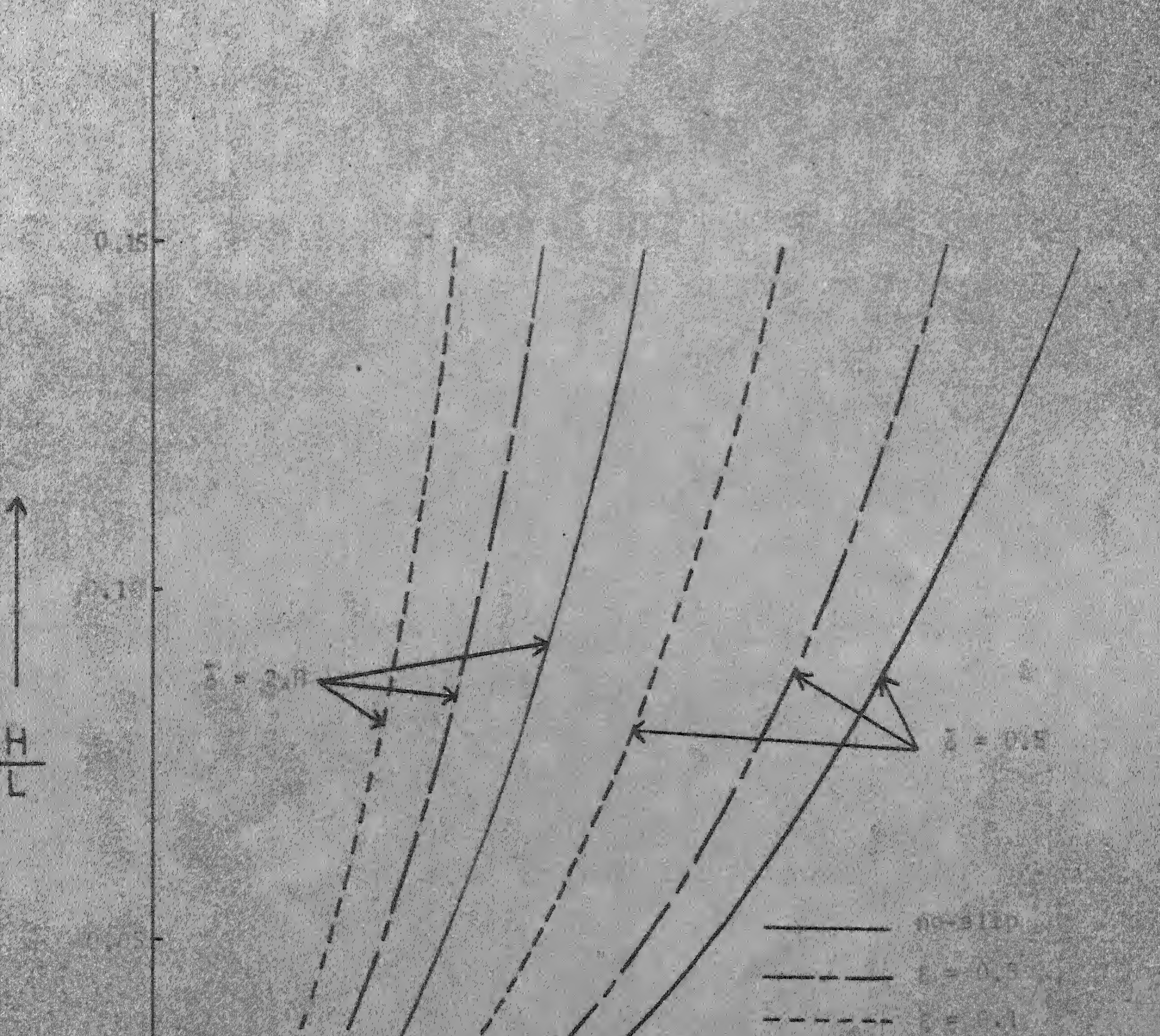

Fig. 2.10 POROUS THICKNESS AND TIME RELATION

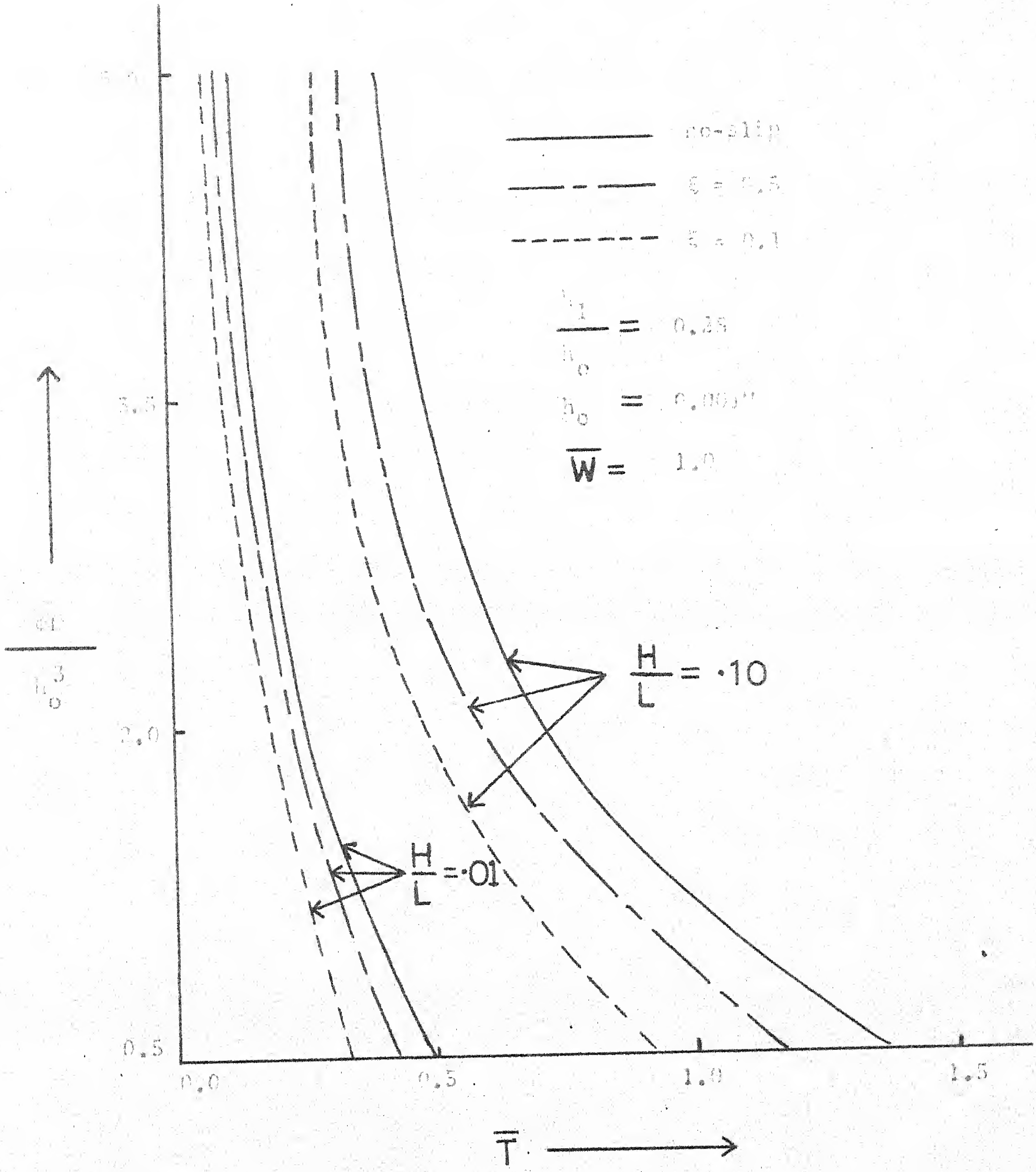

Fig. 2.11 PERMEABILITY PARAMETER AND TIME RELATION

2.7 RESULTS

In this chapter, a generalized form of Reynolds equation applicable to fluid (gas as well as liquid) film lubrication is derived by considering the variation of fluid properties across as well as along the film thickness with slip velocities at the bearing surfaces. Various particular forms are discussed.

In case (I), a Reynolds equations with slip is derived for compressible gas lubricant and the case of an externally pressurised bearing is discussed analytically. It has been shown that the pressure in the gas film decreases as the mean free path increases.

In Case (II), three different layers of fluid with different viscosities have been considered and a Reynolds equation with slip for two dimensional slider bearing is deduced. In one dimensional case, it has been shown that the load capacity and frictional force increase as the slip coefficient and the viscosities of the fluid layers increase. The same result has been pointed out in the case of an externally pressurised bearing. [See case (III)]

In general it has been noted that the effect of slip at the bearing surfaces is to decrease the load capacity and frictional force.

In case (IV), the slip velocities are considered at the porous surfaces to obtain ageneralized Reynolds equation for incompressible lubricant. Even, in the case of externally pressurized porous squeeze bearing, the existant of slip velocity at the porous surface reduces the load capacity and shorten the time of squeezing.

LIST OF NOTATIONS

h	-	film thickness
h_o	-	initial film thickness
h_1	-	final film-thickness
H_1, H_2	-	distance between the surfaces and the origion.
L	-	length of bearing
p	-	pressure
p_i,p_a	-	inlet pressure
p_e	-	external pressure
r	-	radial coordinate
r_i	-	inlet radious
r_e	-	outlet radious
t	-	time of squeezing
u,v,w	-	velocity components of the film in x, y and z directions
U_1,U_2	-	velocities of the surfaces at $z = H_1$ and $z = H_2$, in x direction
V_1,V_2	-	velocities of the surface at $z = H_1$ and $z = H_2$, in y direction
V	-	Squeeze velocity
W	-	load capacity
x,y,z	-	rectangular coordinate
ρ	-	density of the lubricant
η,η_1, η_2	-	viscosities of the lubricant
$\lambda_1,\lambda_2,\delta_1,\delta_2$	-	parameters defined after equation (2.8)
ϕ	-	permeable parameter of porous facing
$(\ )_1$	-	value at $z = H_1$
$(\ )_2$	-	value of $z = H_2$

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CHAPTER - III

A THEORY OF STOCHASTIC LUBRICATION

3.1 INTRODUCTION :

Owing to machining limitations, it is not possible to manufacture perfectly smooth surfaces of the bearings. Naturally, therefore the question arises as to what happens to the various characteristics of the bearing when the surfaces are rough. This question has been taken up by some workers Burton [1963] , Shukla and Prasad [1966], Dowson and Whomes [1971] in the past assuming that rough surface to be represented by a single sine (cosine) wave or a series of sine (cosine) waves and thus modifying the film thickness in the usual study of the bearing characteristics. It has been pointed out that the load capacity, frictional force etc. can be different from their value corresponding to the smooth surface and this difference depends mainly on the amplitudes and the wave lengths of the waves representing the rough surface. This procedure, called the deterministic approach has also been applied to study the characteristics of rollers Dowson and Whomes [1971] and spiral groove bearings Wildmann [1968] .

In another method, called the stochastic approach, the surface roughness is assumed to be represented by a stochastic process

and the usual procedure of study is followed by taking the statistical mean of the basic equation. In fact, it has been demonstrated that the density distribution of roughness heights in usually machined surfaces follow nearly the gaussian distribution and the gap between two rough surfaces can be represented by a stochastic process Papoulis [1965]. Using this approach the characteristics of an infinite slider and short journal bearings has been studied Tzeng and Saibel [1967a, 1967b] . The Reynolds equation applicable to finite rough bearings has also been derived under certain assumptions, Christensen [1969-70] , Christensen and Tonder [1969a] and the case of finite slider bearing has been investigated, Christensen and Tonder [1970] . This concept has also been extended to rough bearings using compressible lubricants, Elord [1973].

In this chapter, we derive a generalized form of Reynolds equation applicable to finite rough bearings by using stochastic approach, Sveshnikov [1966] and Lin [1967] The cases of hydrodynamic one dimensional step slider bearing and hydrostatic bearing are investigated.

3.2 BASIC EQUATIONS

Consider the flow of an incompressible lubricant between a stationary rough surface and a moving smooth surface (see Fig. 3.1). The equation governing the pressure in the film is given by,

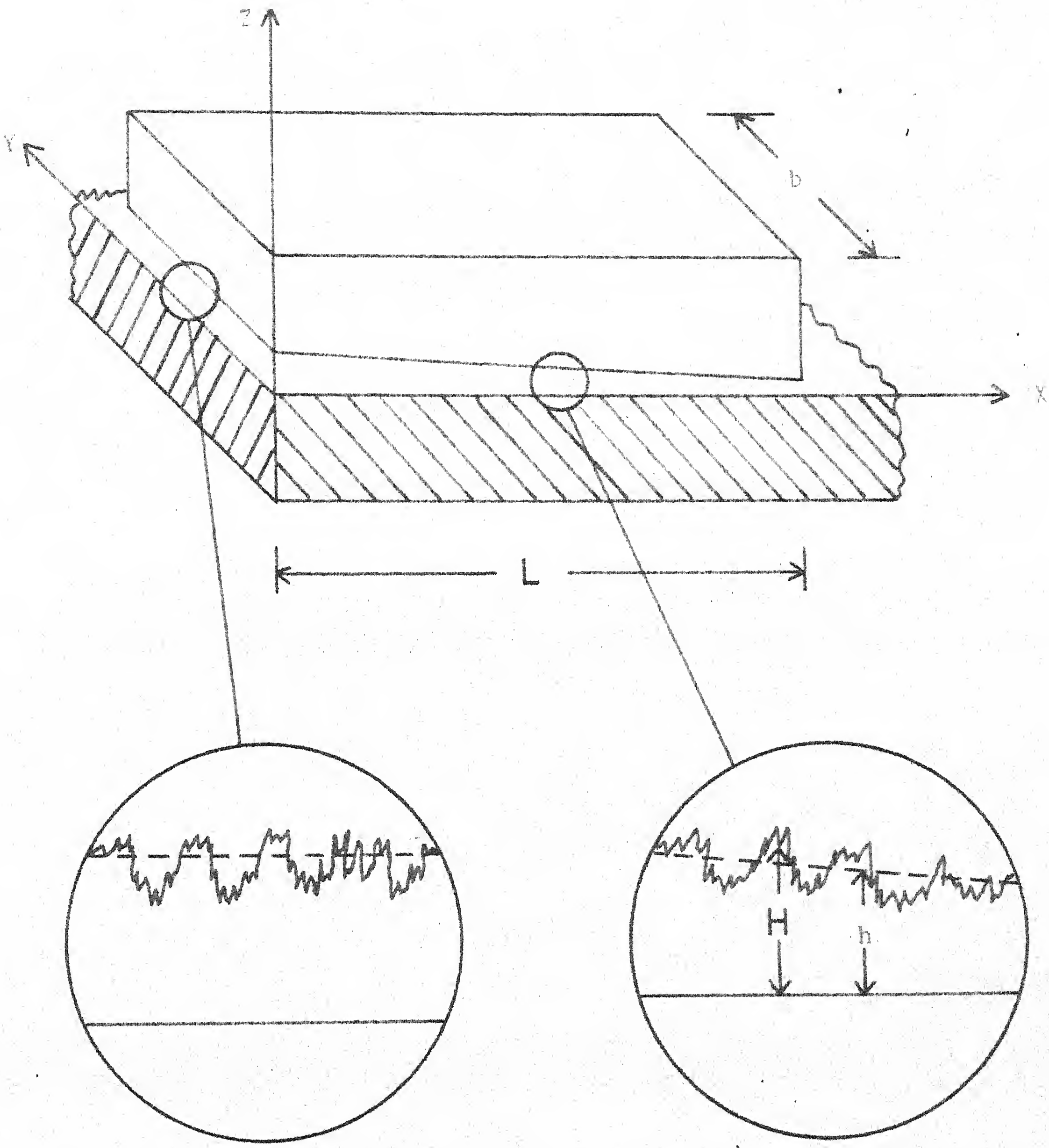

Fig. 3.1 TOPOGRAPHY OF SURFACES UNDER LUBRICATION

$$\frac{\partial}{\partial x}\left(H^3 \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(H^3 \frac{\partial P}{\partial y}\right) = 6\eta U \frac{\partial H}{\partial x} + 6\eta V \frac{\partial H}{\partial y} \tag{3.1}$$

where the pressure P can, in principle, be determined for a given film thickness H. Since the film thickness function H is a stochastic process, the pressure also becomes a stochastic process and can only be determined by an averaging process, Peklenik [1968] and Papoulis [1965]. Thus, for a given random input H, equation (3.1) is a stochastic differential equation for determining the random function P, Christensen [1969-70].

Now, by taking the statistical average or expected value of each terms of equation (3.1) we have,

$$\frac{\partial}{\partial x} E\left(H^3 \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y} E\left(H^3 \frac{\partial P}{\partial y}\right) = 6\eta U \frac{\partial}{\partial x} E(H) + 6\eta V \frac{\partial}{\partial y} E(H) \tag{3.2}$$

where

$E(s) = \int_{-\infty}^{\infty} sF(s)\, ds$ and $F(s)$ is the probability density distribution of the stochastic variable s, H being the film-thickness function, Tzeng and Saibel [1967-a] and Christensen[1969-70] written as follows :

$$H = h(x,y) + h_s(x,y,\xi) \tag{3.3}$$

Here h is the nominal film-thickness (the film-thickness corresponding to smooth surfaces) and h_s is the part of the film-thickness due to surface roughness measured from the nominal level and is assumed to be a (stationary) ergodic, stochastic process.

In general, two stochastic processes P and H are correlated and even if all the statistical characteristics of H are known, it is not possible to find out the mean of such terms as $H^3 \frac{\partial P}{\partial x}$ in terms of the means of H and $\frac{\partial P}{\partial x}$ explicitly unless some extra relations are assumed, Christensen [1969-70], Christensen and Tonder [1969a, 1969b]. As such, since a stochastic function can be approximated as a series expension in terms of deterministic and dependent or independent stochastic functions Sveshnikov [1966] and Lin [1967], it is assumed that the flow fluxes which are stochastic processes can be written in terms of H as follows :

$$Q_x = \frac{U}{2} H - \frac{1}{12\eta} H^3 \frac{\partial P}{\partial x} = \frac{U}{2} \sum_{i=0}^{M} A_i H^i \tag{3.4-a}$$

$$Q_y = \frac{V}{2} H - \frac{1}{12\eta} H^3 \frac{\partial P}{\partial y} = \frac{V}{2} \sum_{j=0}^{N} B_j H^j \tag{3.4-b}$$

Here A_i's and B_j's are non-random functions of x,y,U and V or constants depending upon the type of roughness, roughness slopes, minimum film-thickness etc. and may be determined theoretically or experimentaly [see appendix IV]. However, it will be seen later that by choosing A_i's and B_j's suitably, some well known cases can be obtained, Tzeng and Saibel [1967-a], Christensen [1969-70], Christensen and Tonder [1969-b, 1970].

In general, as pointed out by Tonder and Christensen [1972], the flux at a point in a lubricant film may not be determined by the local film-thickness alone such as in the case of waviness (long wave undulations) where h_s is comparable to the nominal film-thickness.

However, in the case of roughness i.e. the case of high frequency undulations with smaller amplitudes (i.e. $h_s \ll h$) the representations (3.4) are applicable. Further, it may also assume that the mean of h_s is zero [i.e. $E(h_s) = 0$] and then $E(H) = h$.

Taking the expected value of equation (3.4-a) we have

$$\frac{U}{2} E(H) - \frac{1}{12\eta} E(H^3 \frac{\partial P}{\partial x}) = \frac{U}{2} \sum_{i=0}^{M} A_i E(H^i) \tag{3.5}$$

Again, dividing equation (3.4-a) by H^3 and taking the expected value, we get,

$$\frac{U}{2} E(\frac{1}{H^2}) - \frac{1}{12\eta} E(P) = \frac{U}{2} \sum_{i=0}^{M} A_i E(H^{i-3}) \tag{3.6}$$

Eliminating A_m where $0 \leq m \leq M$ from equations (3.5) and (3.6), we have

$$\frac{U}{2} E(H) - \frac{1}{12\eta} E(H^3 \frac{\partial P}{\partial x}) = \frac{U}{2} [\sum_{i=0, i\neq m}^{M} A_i E(H^i) + \frac{E(H^m)}{E(H^{m-3})} \times \{ E(\frac{1}{H^2}) - \sum_{i=0, i\neq m}^{M} A_i E(H^{i-3}) \}] - \frac{1}{12\eta} \frac{E(H^m)}{E(H^{m-3})} \frac{\partial}{\partial x} [E(P)] \tag{3.7-a}$$

Similarly, from equation (3.4-b) we get,

$$\frac{V}{2} E(H) - \frac{1}{12\eta} E(H^3 \frac{\partial P}{\partial y}) = \frac{V}{2} [\sum_{j=0, j\neq n}^{N} B_j E(H^j) + \frac{E(H^m)}{E(H^{n-3})} \times \{ E(\frac{1}{H^2}) - \sum_{j=0, j\neq n}^{N} B_j E(H^{j-3}) \}] - \frac{1}{12\eta} \frac{E(H^n)}{E(H^{n-3})} \frac{\partial}{\partial y} [E(P)] \tag{3.7-b}$$

Now from equations (3.2), (3.7-a) and (3.7-b) the generalized form of Reynolds equation, applicable to a finite bearing of which the moving surface is smooth, can be written as follows :

$$\frac{\partial}{\partial x}\left[\frac{E(H^m)}{E(H^{m-3})}\frac{\partial}{\partial x}E(P)\right]+\frac{\partial}{\partial y}\left[\frac{E(H^n)}{E(H^{n-3})}\frac{\partial}{\partial y}E(P)\right]$$

$$=6\eta U\frac{\partial}{\partial x}\left[\sum_{i=0,i\neq m}^{M}A_i\,E(H^i)+\frac{E(H^m)}{E(H^{m-3})}\left\{E\left(\frac{1}{H^2}\right)-\sum_{i=0,i\neq m}^{M}A_i E(H^{i-3})\right\}\right]$$

$$+6\eta V\frac{\partial}{\partial y}\left[\sum_{j=0,j\neq n}^{N}B_j\,E(H^j)+\frac{E(H^n)}{E(H^{n-3})}\left\{E\left(\frac{1}{H^2}\right)-\sum_{j=0,j\neq n}^{N}B_j E(H^{j-3})\right\}\right]$$

(3.8)

Finally, equation (3.8) can be written in a more simplified form thus :

$$\frac{\partial}{\partial x}\left[h^3 f_m\frac{\partial}{\partial x}E(P)\right]+\frac{\partial}{\partial y}\left[h^3 f_n\frac{\partial}{\partial y}E(P)\right]$$

$$=6\eta U\frac{\partial}{\partial x}(g_m)+6\eta V\frac{\partial}{\partial y}(g_n) \qquad (3.8\text{-A})$$

where

$$f_m=\frac{E(H^m)}{h^3\,E(H^{m-3})},$$

$$g_m=\left[\sum_{i=0,i\neq m}^{M}A_i E(H^i)+f_m h^3\left\{E\left(\frac{1}{H^2}\right)-\sum_{i=0,i\neq m}^{M}A_i\,E(H^{i-3})\right\}\right]$$

and similar expressions for f_n and g_n. The functions f_m, f_n, g_m and g_n, would have different functional relation with $E(H^i)$ for different types of roughness and indices m,n may depend upon roughness heights, roughness slopes etc. The solution of equation (3.8-A) can be

easily plotted for different values of these functions in tabular form for convenient use.

Further, in the case of compressible lubricant, if ρ is assumed to be function of the mean pressure $E(P)$ only i.e. $\rho = \rho\{E(P)\}$ and its variance is neglected, then by following the above procedure, the corresponding Reynolds equation for gas lubricated bearing can be deduced in the first approximation as follows :

$$\frac{\partial}{\partial x}\left[\frac{E(H^m)}{E(H^{m-3})}\ \rho\{E(P)\}\frac{\partial}{\partial x}E(P)\right] + \frac{\partial}{\partial y}\left[\frac{E(H^n)}{E(H^{n-3})}\rho\{E(P)\}\frac{\partial}{\partial y}E(P)\right]$$

$$= 6\eta U\frac{\partial}{\partial x}\left[\sum_{i=0,i\neq m}^{M} A_i E(H^i) + \frac{E(H^m)}{E(H^{m-3})}\{E(\frac{1}{H^2})\ \rho\{E(P)\} - \sum_{i=0,i\neq m}^{M} A_i E(H^{i-3})\}\right]$$

$$+ 6\eta V\frac{\partial}{\partial y}\left[\sum_{j=0,j\neq n}^{N} B_j E(H^j) + \frac{E(H^n)}{E(H^{n-3})}\{E(\frac{1}{H^2})\rho\{E(P)\} - \sum_{j=0,j\neq n}^{N} B_j E(H^{j-3})\}\right]$$

(3.8-B)

Here A_i's and B_j's represent the various type of roughness, roughness distribution and roughness correlation of the bearing surface.

Now in the following some particular cases are considered by taking $\underset{j\neq n}{B_j} = 0$ for all j and V = 0.

3.3 PARTICULAR CASES

CASE (i). By considering $\underset{i\neq m}{A_i} = 0$ for all i in equation (3.8), we get the Reynolds type equation applicable to correlated roughness along sliding direction as follows :

$$\frac{\partial}{\partial x}\left[\frac{E(H^m)}{E(H^{m-3})}\frac{\partial}{\partial x}E(P)\right]+\frac{\partial}{\partial y}\left[\frac{E(H^n)}{E(H^{n-3})}\frac{\partial}{\partial y}E(P)\right]$$

$$= 6\eta U\frac{\partial}{\partial x}\left[\frac{E(H^m)}{E(H^{m-3})}E\left(\frac{1}{H^2}\right)\right] \tag{3.8-C}$$

(i-1) If m = 0 and n = 3, we have from equation (3.8-C)

$$\frac{\partial}{\partial x}\left[\frac{1}{E\left(\frac{1}{H^3}\right)}\frac{\partial}{\partial x}E(P)\right]+\frac{\partial}{\partial y}\left[E(H^3)\frac{\partial}{\partial y}E(P)\right]$$

$$= 6\eta U\frac{\partial}{\partial x}\left[\frac{E\left(\frac{1}{H^2}\right)}{E\left(\frac{1}{H^3}\right)}\right] \tag{3.8-D}$$

which is a Reynolds type equation for non-uniform, transverse roughness, correlated along sliding direction (one-dimensional transverse roughness) Christensen and Tonder [1970].

In one-dimensional case the above equation reduces to the following form which is applicable to an infinite slider bearing, Tzeng and Saibel [1967-a].

$$\frac{d}{dx}\left[\frac{1}{E\left(\frac{1}{H^3}\right)}\frac{d}{dx}E(P)\right] = 6\eta U\left[\frac{E\left(\frac{1}{H^2}\right)}{E\left(\frac{1}{H^3}\right)}\right] \tag{3.8-E}$$

(i-2) If m = 3 and n = 0, then from equation (3.8-C),

$$\frac{\partial}{\partial x}\left[E(H^3)\frac{\partial}{\partial x}E(P)\right]+\frac{\partial}{\partial y}\left[\frac{1}{E\left(\frac{1}{H^3}\right)}\frac{\partial}{\partial y}E(P)\right]$$

$$= 6\eta U\frac{\partial}{\partial x}\left[E(H^3)\,E\left(\frac{1}{H^2}\right)\right] \tag{3.8-F}$$

which is applicable for non-uniform, longitudinal roughness, correlated along sliding direction.

(i-3) If $m = 3$ and $n = 3$, then from equation (3.8-C), we have

$$\frac{\partial}{\partial x}\left[E(H^3)\frac{\partial}{\partial x}E(P)\right] + \frac{\partial}{\partial y}\left[E(H^3)\frac{\partial}{\partial y}E(P)\right] = 6\eta U\frac{\partial}{\partial x}\left[E(H^3)\,E\left(\frac{1}{H^2}\right)\right] \qquad (3.8\text{-G})$$

which is a Reynolds type equation for uniform, longitudinal roughness, correlated along sliding direction.

(i-4) If $m = 0$ and $n = 0$, then from equation (3.8-C),

$$\frac{\partial}{\partial x}\left[\frac{1}{E\left(\frac{1}{H^3}\right)}\frac{\partial}{\partial x}E(P)\right] + \frac{\partial}{\partial y}\left[\frac{1}{E\left(\frac{1}{H^3}\right)}\frac{\partial}{\partial y}E(P)\right] = 6\eta U\frac{\partial}{\partial y}\left[\frac{E\left(\frac{1}{H^2}\right)}{E\left(\frac{1}{H^3}\right)}\right] \qquad (3.8\text{-H})$$

which is a Reynolds type equation for uniform, transverse roughness, correlated along sliding direction.

CASE (ii) : By considering $A_1 = 1$, $A_i = 0$ for all i in equation (3.8), $i \neq 1, m$

we get the following Reynolds type equation applicable to less-correlated roughness along sliding direction.

$$\frac{\partial}{\partial x}\left[\frac{E(H^m)}{E(H^{m-3})}\frac{\partial}{\partial x}E(P)\right] + \frac{\partial}{\partial y}\left[\frac{E(H^n)}{E(H^{n-3})}\frac{\partial}{\partial y}E(P)\right] = 6\eta U\frac{\partial}{\partial x}\left[E(H)\right] \qquad (3.8\text{-I})$$

(ii-1) : If $m = 0$ and $n = 3$, from above equation, one gets

$$\frac{\partial}{\partial x}\left[\frac{1}{E\left(\frac{1}{H^3}\right)}\frac{\partial}{\partial x}E(P)\right] + \frac{\partial}{\partial y}\left[E(H^3)\frac{\partial}{\partial y}E(P)\right] = 6\eta U\frac{\partial}{\partial x}\left[E(H)\right] \qquad (3.8\text{-J})$$

which is applicable for non-uniform, transverse roughness, uncorrelated along sliding direction.

(ii-2) : If m = 3 and n = 0, from equation (3.8-I), we get

$$\frac{\partial}{\partial x}\left[E(H^3)\frac{\partial}{\partial x}E(P)\right] + \frac{\partial}{\partial y}\left[\frac{1}{E(\frac{1}{H^3})}\frac{\partial}{\partial y}E(P)\right] = 6\eta U\frac{\partial}{\partial x}\left[E(H)\right] \qquad (3.8\text{-K})$$

which is applicable for, non-uniform, longitudinal roughness, un-correlated along sliding direction (one-dimensional longitudinal roughness), Christensen and Tonder [1970].

(ii-3) : If m = 3 and n = 3, from equation (3.8-I), one gets

$$\frac{\partial}{\partial x}\left[E(H^3)\frac{\partial}{\partial x}E(P)\right] + \frac{\partial}{\partial y}\left[E(H^3)\frac{\partial}{\partial y}E(P)\right] = 6\eta U\frac{\partial}{\partial x}\left[E(H)\right] \qquad (3.8\text{-L})$$

which is applicable to uniform, longitudinal roughness, un-correlated along sliding direction (isotropic) as derived by Christensen and Tonder [1969-a].

(ii-4) : If m = 0 and n = 0, from equation (3.8-I), we get

$$\frac{\partial}{\partial x}\left[\frac{1}{E(\frac{1}{H^3})}\frac{\partial}{\partial x}E(P)\right] + \frac{\partial}{\partial y}\left[\frac{1}{E(\frac{1}{H^3})}\frac{\partial}{\partial y}E(P)\right] = 6\eta U\frac{\partial}{\partial x}\left[E(H)\right] \qquad (3.8\text{-M})$$

which is applicable for uniform, transverse roughness, un-correlated along sliding direction.

It is seen from the above forms of equations that when m and n are equal, the term 'uniform roughness' has been used, while when they are different, the term 'non-uniform' is suggested. The term

'transverse roughness' basically implies a correlation between the film-thickness function and the pressure gradient in the x-direction while the term longitudinal roughness suggests non-correlation in the above terms. The correlation or otherwise between different film-thickness functions in the sliding velocity term has been indicated in various cases by writing these relations in the "sliding direction".

It can also be noted that the roughness correlation decreases as m and n deviate from zero to three and as A_1 increases from zero to unity.

In the following, the two particular cases of bearings are studied where norminal film-thickness is constant.

(I) STEP BEARING

(II) HYDROSTATIC BEARING

3.4 STEP BEARING

Consider an infinite rough step bearing as shown in Fig. (3.2). Since, for all practical purpose $E(H_1^m)$ and $E(H_2^m)$ can be assumed constants, we have from equation (3.8) the equation determining the pressure as follows :

$$\frac{\partial}{\partial x} E(P_i) = \text{const.} \qquad \begin{array}{l} i = 1 \text{ in region I} \\ i = 2 \text{ in region II} \end{array} \tag{3.9}$$

Since the pressure at the step is continuous, we have,

$$E(P_s) = kL \frac{\partial}{\partial x} E(P_1) = -(L - kL) \frac{\partial}{\partial x} E(P_2) \tag{3.10}$$

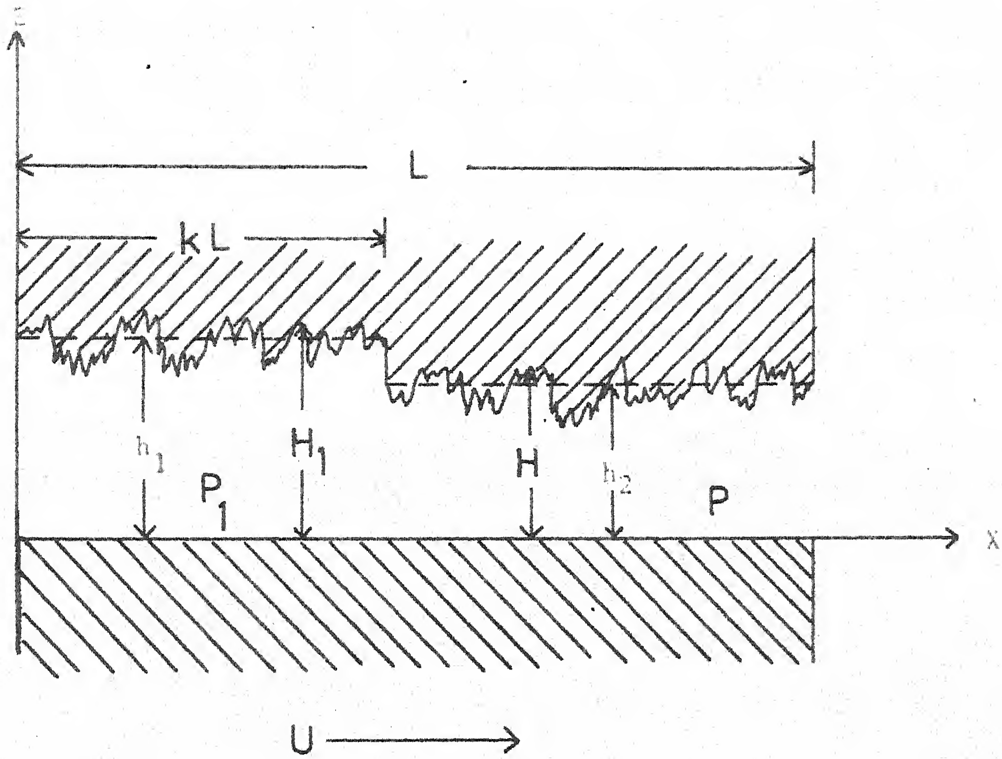

Fig. 3.2 ROUGH STEP SLIDER BEARING

Using the continuity of flow fluxes in the two regions and remembering equation (3.7-a) we can write :

$$E(Q) = -\frac{b}{12\eta}\frac{E(H_1^m)}{E(H_1^{m-3})}\frac{\partial}{\partial x}E(P_1) + \frac{Ub}{2}\Big[\sum_{i=0,i\neq m}^{M} A_i E(H_1^i) + \frac{E(H_1^m)}{E(H_1^{m-3})}\times$$

$$\Big\{E\Big(\frac{1}{H_1^2}\Big) - \sum_{i=0,i\neq m}^{M} A_i E(H_1^{i-3})\Big\}\Big]$$

$$= -\frac{b}{12\eta}\frac{E(H_2^m)}{E(H_2^{m-3})}\frac{\partial}{\partial x}E(P_2) + \frac{Ub}{2}\Big[\sum_{i=0,i\neq m}^{M} A_i E(H_2^i) + \frac{E(H_2^m)}{E(H_2^{m-3})}\times$$

$$\Big\{E\Big(\frac{1}{H_2^2}\Big) - \sum_{i=0,i\neq m}^{M} A_i E(H_2^{i-3})\Big\}\Big] \qquad (3.11)$$

Solving equations (3.10) and (3.11) and integrating the resulting expressions under usual boundary conditions, the following equations for determining the pressure is obtained.

$$E(P_1) = \frac{6U\eta\,(1-k)\,(g_1 - g_2)\,x}{(1-k)\,f_1 h_1^3 + k f_2\,h_2^3} \quad 0 \leq x \leq kL \qquad (3.12)$$

and

$$E(P_2) = \frac{6U\eta\,k\,(g_1 - g_2)\,(L-x)}{(1-k)\,f_1 h_1^3 + k\,f_2\,h_2^3} \quad kL \leq x \leq L \qquad (3.13)$$

where

$$f_\alpha = \frac{E(H_\alpha^m)}{h_\alpha^3\,E(H_\alpha^{m-3})},$$

$$g_\alpha = \sum_{i=0,i\neq m}^{M} A_i E(H_\alpha^i) + f_\alpha h_\alpha^3\Big\{E\Big(\frac{1}{H_\alpha^2}\Big) - \sum_{i=0,i\neq m}^{M} A_i E(H_\alpha^{i-3})\Big\}, \alpha=1,2$$

$$(3.14)$$

Proceeding in the usual manner, we write the following expressions for the flow flux and the load capacity respectively as follow :

$$E(Q) = \frac{bU}{2}\left[g_1 - \frac{(1-k)(g_1-g_2)}{(1-k)+k\frac{f_2h_2^3}{f_1h_1^3}}\right] \tag{3.15}$$

$$E(W) = 3\eta UbL^2 \frac{k(1-k)\,(g_1 - g_2)}{(1-k)f_1h_1^3 + kf_2h_2^3} \tag{3.16}$$

The frictional force on the moving surface can be written similarly as

$$E(F) = \eta bUL\left[\frac{3k(1-k)(g_1-g_2)(f_1'h_1 - f_2'h_2)}{(1-k)f_1h_1^3 + kf_2h_2^3} + \{kg_1' + (1-k)g_2'\}\right] \tag{3.17}$$

where

$$f_\alpha' = \frac{E(H_\alpha^{m-2})}{h_\alpha\, E(H_\alpha^{m-3})},$$

$$g_\alpha' = E\left(\frac{1}{H_\alpha}\right) + 3\left[E\left(\frac{1}{H_\alpha}\right) - \sum_{i=0,i\neq m}^{M} A_i\, E(H_\alpha^{i-2})\right.$$

$$\left. - f_\alpha'h_\alpha\left\{E\left(\frac{1}{H_\alpha^2}\right) - \sum_{i=0,i\neq m}^{M} A_i\, E(H_\alpha^{i-3})\right\}\right] \quad , \alpha = 1,2 \tag{3.18}$$

The expressions for f_α , f_α' , g_α and g_α' are expressed in terms of σ/h_2 for various values of m and A_i's in Appendix I.

The friction-coefficient can be defined as follows :

$$E(\mu) = \frac{E(F)}{E(W)} \tag{3.19}$$

The expression for $E(W)$, $E(F)$ and $E(\mu)$ are plotted in Figs. (3.3), (3.4) and (3.5) respectively for the following combinations of m and A_i's ($\frac{h_1}{h_2} = 1.866$, $k = 0.7$) for different values of σ/h_2.

(a) $\left.\begin{array}{l} A_1 = 0 \\ \quad \text{or} \\ A_1 = 1 \end{array}\right]$ for $m = 0$, $\underset{i \neq 0}{A_i} = 0$ for all i

(b) $\left.\begin{array}{l} A_1 = 0 \\ \quad \text{or} \\ A_1 = 1 \end{array}\right]$ for $m = 3$, $\underset{i \neq 3}{A_i} = 0$ for all i.

Hence, $m = 0$ corresponds to the transverse roughness while $m = 3$ corresponds to the longitudinal roughness. The condition $A_1 = 0$ implies a correlation between the film-thickness functions in the sliding velocity term while $A_1 = 1$ corresponds a non-correlation in these functions. Physically, for the same asperity height, the case $A_1 = 0$ corresponds to sharp and rough asperity slope with small asperity base while $A_1 = 1$ corresponds to wavy, less sharp, smooth asperity slope with comparatively large base.

From the figures (3.3), (3.4) and (3.5) it can be seen that in case (a) the load capacity, friction force increase but the coefficient of friction decreases as σ/h_2 increases. It can also be seen that the increase in load capacity is more significant than that of friction force and this leads to a lower coefficient of friction. These results are similar to that obtained by

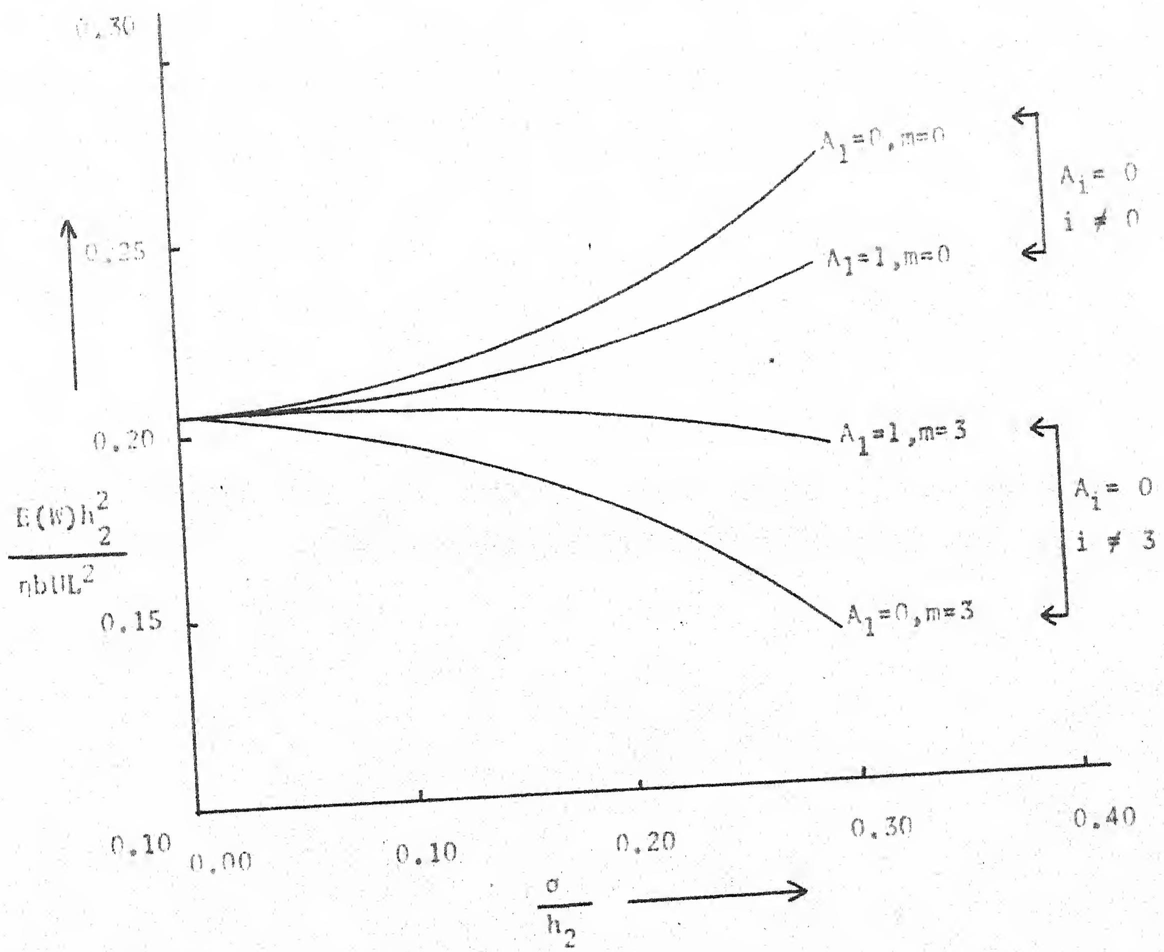

Fig. 3.3 LOAD CAPACITY ($k = 0.7$, $\frac{h_1}{h_2} = 1.866$)

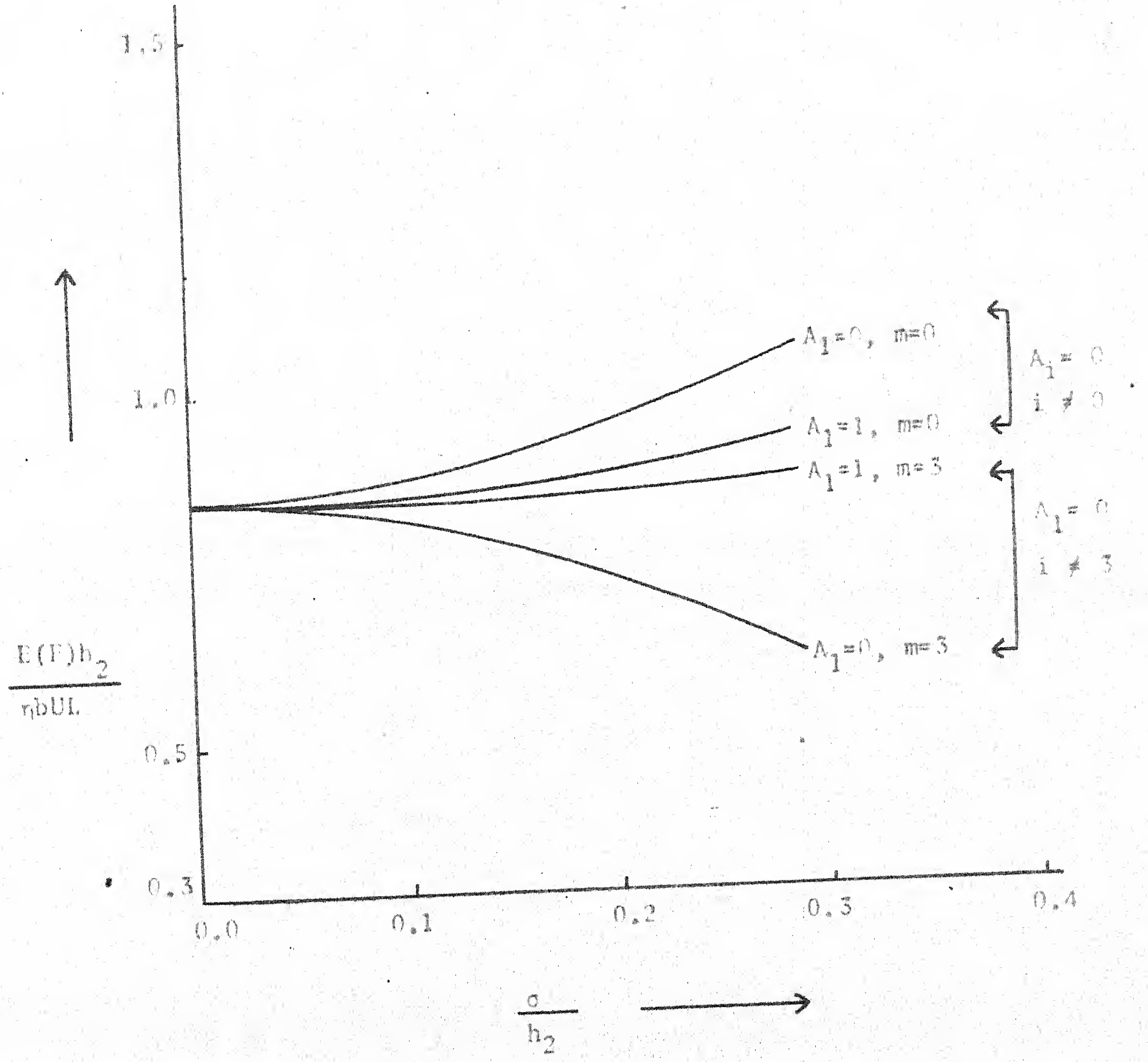

Fig. 3.4 FRICTION FORCE ($k = 0.7$, $\frac{h_1}{h_2} = 1.866$)

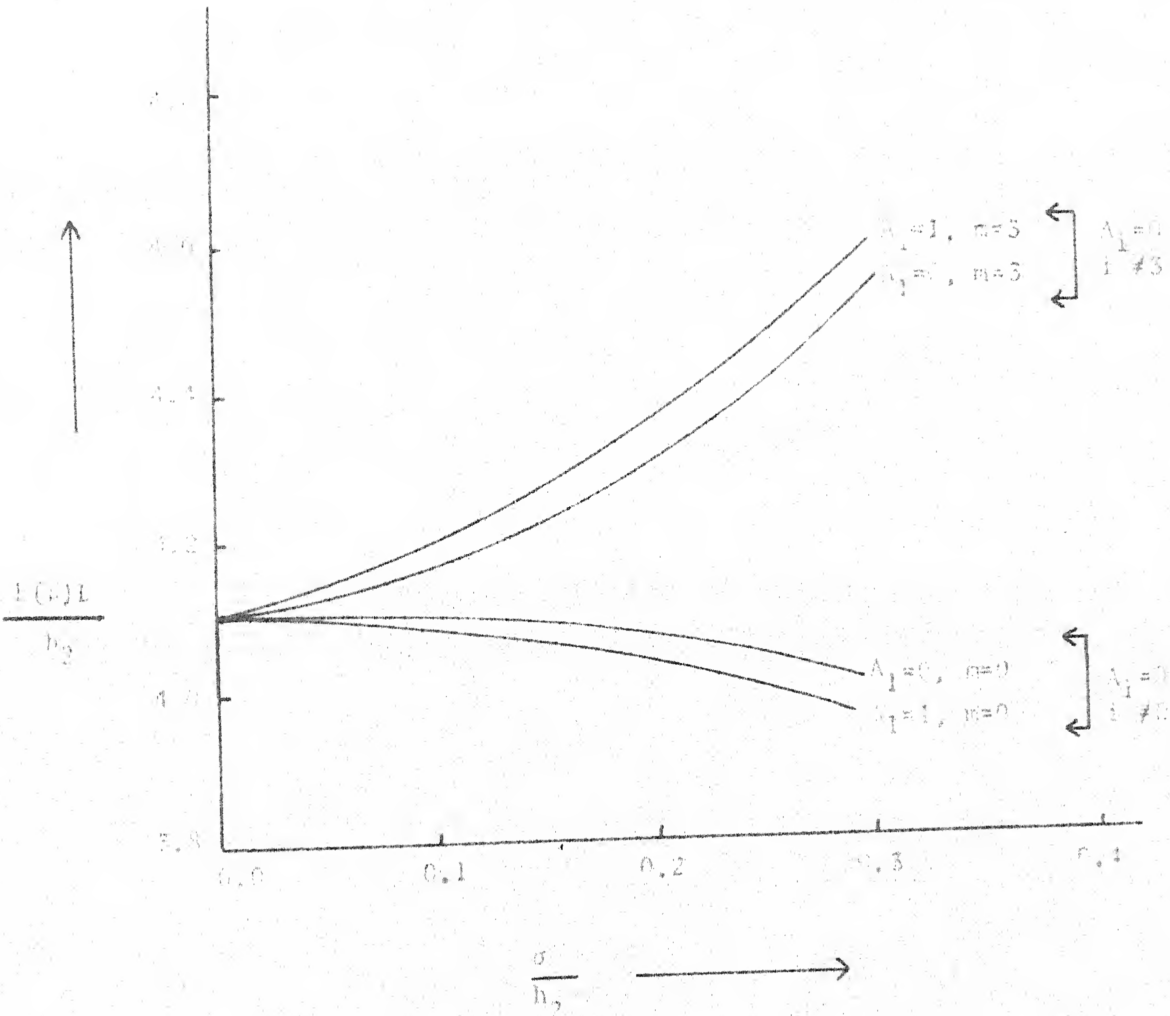

Fig. 3.5 FRICTION COEFFICIENT ($k=0.7$, $\frac{h_1}{h_2} = 1.866$)

Tzeng and Saibel [1967-a] for rough slider bearing (i.e $m = 0$, $A_1 = 0$, $\underset{i \neq 0}{A_i} = 0$ for all i). It can also be seen that the load capacity in friction force are greater in the case of a step slider bearing than the inclined slider case, for the same roughness and other characteristics, Tzeng and Saibel [1967-a] .

Further when $A_1 = 0$, the increase in load capacity of the step slider bearing in comparison to smooth step bearing is about 35% for $\sigma/h_2 = 0.3$ while the decrease in coefficient of friction is about 4%.

To see the effect of A_1 in case (a) it is noted from Figs. (3.3), (3.4) and (3.5) that the load capacity, friction force and the coefficient of friction are greater in the case $A_1 = 0$ as compared to the case $A_1 = 1$ for given σ/h_2.

In case (b), these results are opposite to that of case (a) in all cases except the case $A_1 = 1$, $m = 3$, where the change in friction force is not appreciable with respect to the increase in σ/h_2.

3.5 HYDROSTATIC BEARING

Now consider the case of a rough hydrostatic bearing with constant nominal film thickness as shown in Fig. (3.6). Assuming that the variance of the film-thickness is constant we have from equation (3.8) the equation determining the pressure in this case as follows :

$$\frac{\partial}{\partial r}\left[r \frac{E(H^m)}{E(H^{m-3})} \frac{\partial}{\partial r} E(P) \right] = 0 \qquad (3.20)$$

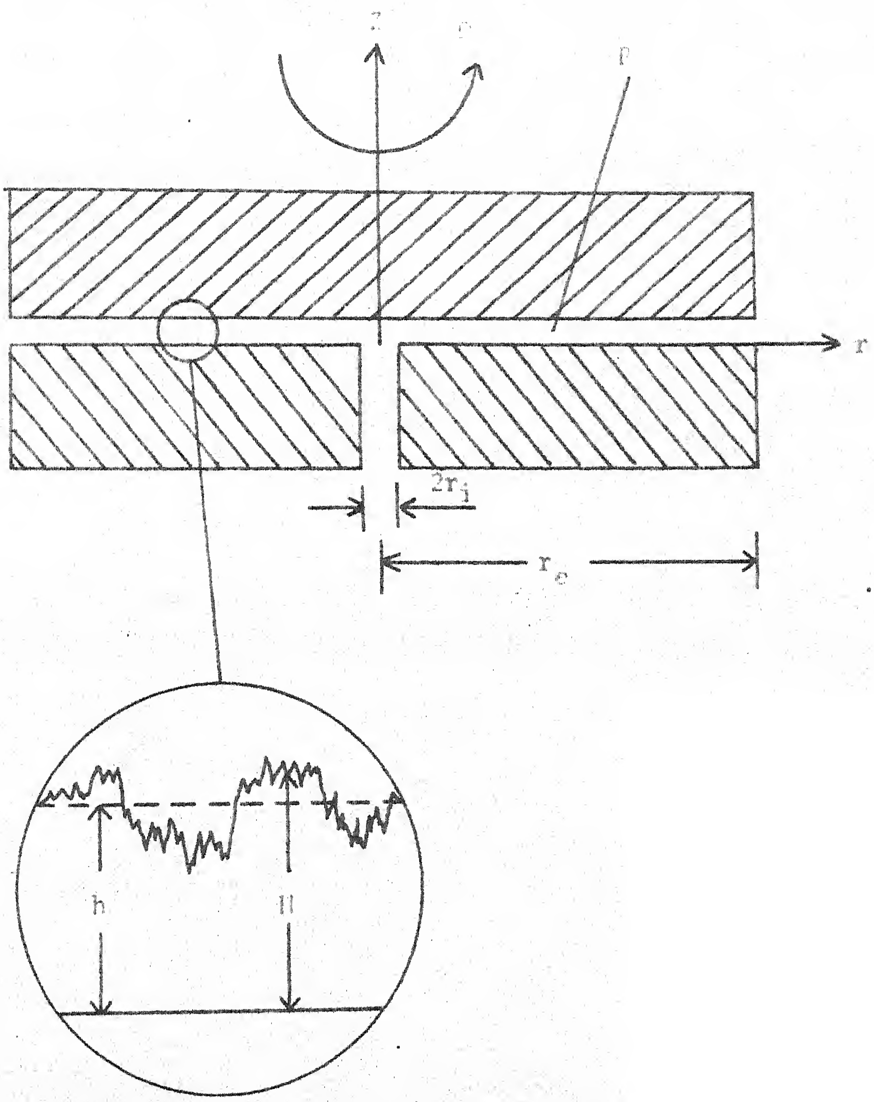

Fig. 3.6 ROUGH HYDROSTATIC BEARING

Integrating equation (3.20) and using the definition of flow flux,

$$E(Q) = -\frac{\pi r}{6\eta}\frac{E(H^m)}{E(H^{m-3})}\frac{\partial}{\partial r}E(P) \tag{3.21}$$

Solving equation (3.21) and using boundary conditions $E(P) = E(P_i)$ at $r = r_i$ and $E(P) = 0$ at $r = r_e$, the expressions for pressures can be written as follows :

$$E(P) = \frac{6\eta E(Q)}{\pi h^3 f_o} \ln \frac{r_e}{r} \tag{3.22}$$

$$E(P_i) = \frac{6\eta E(Q)}{\pi h^3 f_o} \ln \frac{1}{\lambda} \tag{3.23}$$

where

$$f_o = \frac{E(H^m)}{h^3 \; E(H^{m-3})} \quad \text{and} \quad \lambda = \frac{r_i}{r_e}$$

The load capacity in this case is given by

$$E(W) = \frac{3\eta E(Q) r_e^2}{h^3} (1 - \lambda^2) \frac{1}{f_o} \tag{3.24}$$

The behaviour of the function $\frac{1}{f_o}$ is shown in the Fig. (3.7). From this, we see that this function increases for $m = 0$ and decreases for $m = 3$ as σ increases. Thus, we infer from equation (3.21) that the flow flux, at a given inlet mean pressure $E(P_i)$, increases for $m = 3$ and decreases for $m = 0$ with the increase of σ. But if $E(Q)$ is assumed to be constant, then the mean pressure $E(P)$ decreases for $m = 3$ and increases for $m = 0$ as σ increases. Similarly, for a given mean flow flux, the load capacity decreases for $m = 3$ and increases for $m = 0$.

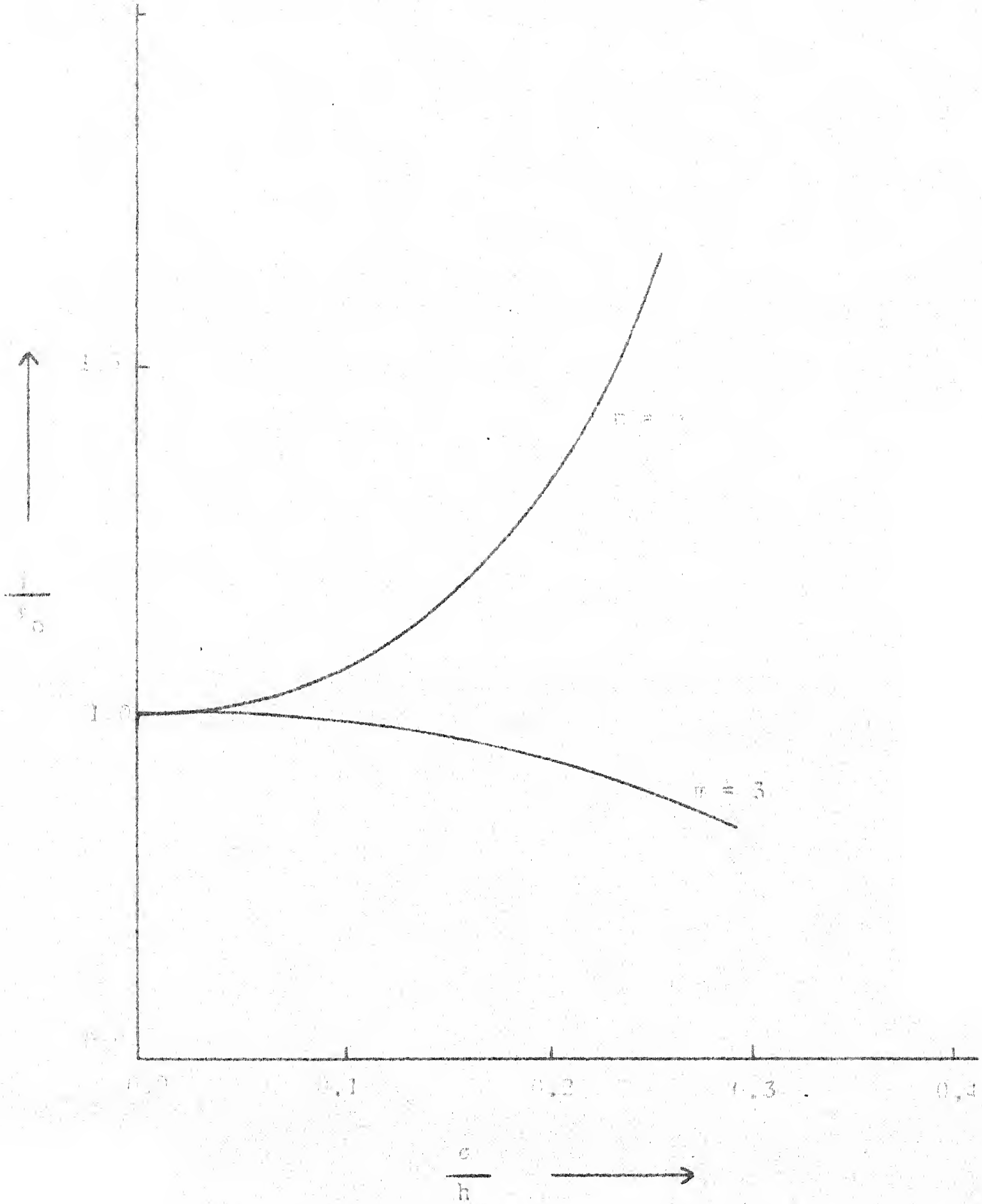

Fig. 3.7 BEHAVIOUR OF THE FUNCTION $\frac{f}{f_o}$

We have also studied the characteristic of this bearing by representing the roughness as a series of cosine function [appendix II] for $\frac{\sigma}{h} \ll 1$. All the characteristics obtained in this section for m = 0, are qualitatively same as that of obtained in appendix II. But if the variance σ is chosen such that $\sigma = \sum_{j=1}^{M} \frac{\varepsilon_j^2}{2}$, then all the characteristics are exactly the same quantitatively as well [For example compare equations (3.24) and (3.46) for $\frac{\sigma}{h} \ll 1$ and the appendix I].

The frictional torque, which is defined by

$$E(T) = \frac{\pi\eta\Omega r_e^4}{2} (1 - \lambda^4) \; E(\tfrac{1}{H}) \qquad (3.25)$$

also increases as σ increases as the function $E(\frac{1}{H})$ always increases with σ .

Finally for $\frac{\sigma}{h} = \cdot 2$, m = 0, the function $\frac{1}{f_0}$ increases by about 30% [see Fig. (3.7)] , hence the load capacity also [see equation (3.24)] increases by the same amount.

3.6 DISCUSSION

In this chapter, the generalized Reynolds equation applicable to rough surfaces has been derived by assuming that the film-thickness function is a stochastic process and satisfies the basic Reynolds equation. To find the average of each term of the stochastic form of Reynolds equation, it has been assumed that the fluxes which are stochastic functions can be represented by power series of the

stochastic variable H, with variable coefficients. From the resulting equation it is seen that, since H is normal, the mean pressure can be expressed only in terms of the variance σ (roughness parameter) as the higher moments of H can be expressed in terms of this variance. By choosing different values of A_i, B_j, m and n which may in general depend upon the type of roughness (i.e. mean, variance, asperity slope, asperity radious, elastic and plastic parameters of the surface etc.), we can derive the various particular cases of the Reynolds equation applicable to the relevant rough bearing surface. From the generalized Reynolds equation (3.8) all the known cases have been derived as particular cases by choosing different values of parameter involved. Further it has been pointed out that one can get the various forms of Reynolds equations applicable to correlated and un-correlated roughness for different values of m,n and A_i's [see CASE (i) and CASE (ii)] .

In the case of rough step bearing it is shown that the load capacity and friction force increase as σ/h_2 increases but the coefficient of friction decreases for the case $m = 0, \; \underset{i\neq 0}{A_i} = 0$

The case of hydrostatic bearings also investigated both by stochastic and sinusoidal approaches[see appendix II]. In both the cases it is shown that the load capacity and the frictional torque increases as the roughness parameter increases for a constant mean inlet pressure, but the flow flux decreases. It is possible by a suitable choice of variance and wave length relation, to make the various characteristics of this bearing exactly the same in the two approaches.

APPENDIX - I : APPROXIMATE DETERMINATION OF FUNCTIONS

In the case of hydrodynamic lubrication, the ratio $\frac{h_s}{h}$ may be assumed to be small ($\frac{\sigma}{h}$ << 1) and we can write

$$H^m = [h + h_s]^m$$

$$\simeq h^m \, [1 + m \, (\frac{h_s}{h}) + \frac{m(m-1)}{2} \, (\frac{h_s}{h})^2 + \ldots] \qquad (3.26)$$

Assuming the density distribution function of h_s to be normal with variance, σ, one can write from equation (3.26) for $\frac{\sigma}{h} << 1$

$$E(H^m) \simeq h^m \quad [1 + \frac{m(m-1)}{2} \frac{\sigma^2}{h^2} + \sigma^4(0)] \qquad (3.27)$$

This function increases as σ increases for m = -1. Using equation (3.27) the function f can be approximated as follows :

$$f_\alpha = \frac{E(H_\alpha^m)}{h_\alpha^3 \; E(H_\alpha^{m-3})} \simeq [1 + 3(m-2) \frac{\sigma^2}{h_\alpha^2} + \sigma^4(0)] \qquad (3.28)$$

This function increases for m = 3 and decreases for m = 0 as σ increases. The function $\frac{f_2}{f_1}$ can also be approximated to the following equation

$$\frac{f_2}{f_1} \simeq [1 + 3(m-2) \frac{(h_1^2 - h_2^2)}{h_1^2 \; h_2^2} \quad \sigma^2 + \sigma^4(0)] \qquad (3.29)$$

Since $h_1 > h_2$ (in the case of step bearing), this function also increases for m = 3 and decreases for m = 0 as σ increases.

Also, the function

$$f'_\alpha = \frac{E(H_\alpha^{m-2})}{h_\alpha \; E(H_\alpha^{m-3})} \simeq [1 + \frac{(m-3)\,\sigma^2}{h_\alpha^2} + \sigma^4(0)] \qquad (3.30)$$

is equal to 1 for m = 3 and it decreases as σ increases for m = 0.

The function

$$f_1' h_1 - f_2' h_2 \simeq (h_1-h_2) [1 - \frac{(m-3)\sigma^2}{h_1 h_2} + \sigma^4(0)] \qquad (3.31)$$

is independent of σ for $m = 3$ and it increases as σ increases for $m = 0$.

Also as the function g_α is,

$$g_\alpha = [\sum_{i=0, i\neq m}^{M} A_i E(H_\alpha^i) + f_\alpha h_\alpha^3 \{E(\frac{1}{H_\alpha^2}) - \sum_{i=0, i\neq m}^{M} A_i E(H_\alpha^{i-3})\}] \qquad (3.32)$$

we have,

$$g_\alpha = h_\alpha \left[\begin{array}{l} \text{(i) } m = 0,\ A_1 = 1,\ \underset{i\neq 0,1}{A_i} = 0 \text{ for all } i, \\ \text{for} \\ \text{(ii) } m = 3,\ A_1 = 1,\ \underset{i\neq 1,3}{A_i} = 0 \text{ for all } i \end{array}\right.$$

and,

$$g_\alpha = \frac{E(\frac{1}{H_\alpha^2})}{E(\frac{1}{H_\alpha^3})} \simeq h_\alpha [1 - \frac{3\sigma^2}{h_\alpha^2} + \sigma^4(0)] ,$$

$$(g_1 - g_2) \simeq (h_1 - h_2) [1 + \frac{3\sigma^2}{h_1 h_2} + \sigma^4(0)] \qquad (3.33)$$

for $m = 0$, $\underset{i\neq 0}{A_i} = 0$ for all i. Thus the function $(g_1 - g_2)$ increases as σ increases.

Again,

$$g_\alpha = E(H_\alpha^3) \cdot E(\frac{1}{H_\alpha^2}) \simeq h_\alpha [1 + 6 \frac{\sigma^2}{h_\alpha^2} + \sigma^4(0)]$$

and

$$(g_1-g_2) \simeq (h_1-h_2) [1 - \frac{6\sigma^2}{h_1 h_2} + \sigma^4(0)] \qquad (3.34)$$

for $m = 3$, $\underset{i \neq 3}{A_i} = 0$ for all i. Thus the function $(\bar{g}_1 - g_2)$ decreases as increases.

Similarly from the function g'_α,

$$g'_\alpha = E(\frac{1}{H_\alpha}) + 3\,[E(\frac{1}{H_\alpha}) - \sum_{i=0, i\neq m}^{M} A_i E(\bar{H}_\alpha^{i-2})$$

$$- f'_\alpha h_\alpha \{E(\frac{1}{H_\alpha^2}) - \sum_{i=0, i\neq m}^{M} A_i E(H_\alpha^{i-3})\}] \qquad (3.35)$$

we have for $m = 0$, $\underset{i \neq 0}{A_i} = 0$ for all i,

$$g'_\alpha = 4E(\frac{1}{H_\alpha}) - \frac{3}{E(\frac{1}{H_\alpha^3})} \{E(\frac{1}{H_\alpha^2})\}^2$$

$$\simeq \frac{1}{h_\alpha}[1 + 4\frac{\sigma^2}{h_\alpha^2} + \sigma^4(0)] \qquad (3.36)$$

which increases as σ increases.

From equation (3.35), for $m = 3$, $\underset{i \neq 3}{A_i} = 0$ for all i, we get,

$$g'_\alpha = 4E(\frac{1}{H_\alpha}) - 3\,E(H_\alpha)\,E(\frac{1}{H_\alpha^2})$$

$$\simeq \frac{1}{h_\alpha}[1 - 5\frac{\sigma^2}{h_\alpha^2} + \sigma^4(0)] \qquad (3.37)$$

which decreases as σ increases.

From equation (3.35),

(i) for $m = 0$, $A_1 = 1$, $\underset{i \neq 0,1}{A_i} = 0$ for all i,

and (ii) for $m = 3$, $A_1 = 1$, $\underset{i \neq 1,3}{A_i} = 0$ for all i,

$$g'_\alpha = E(\frac{1}{H_\alpha}) \simeq \frac{1}{h_\alpha}\ [1 + \frac{\sigma^2}{h_\alpha^2} + \sigma^4(0)] \qquad (3.38)$$

which increases as σ increases.

APPENDIX-II : SINUSOIDAL APPROACH FOR HYDROSTATIC BEARING

In the case of a hydrostatic bearing [see Fig. (3.6)] , the equation determining the pressure is given by

$$\frac{dP}{dr} = - \frac{6\eta Q}{\pi r H^3} \tag{3.39}$$

which, on integrating and using the boundary condition $P = 0$ at $r = r_e$ and $P = P_i$ at $r = r_i$, give

$$P = \frac{6\eta Q}{\pi} \int_{r}^{r_e} \frac{1}{rH^3} \, dr \tag{3.40}$$

and

$$P_i = \frac{6\eta Q}{\pi} \int_{r_i}^{r_e} \frac{1}{rH^3} \, dr \tag{3.41}$$

The equation (3.40) determines the pressure while equation (3.41) gives the flow flux in the bearing.

The load capacity is given by

$$W = \pi P_i \, r_i^2 + \int_{r_i}^{r_e} 2\pi \, rP \, dr$$

which simplifies to the following equation after using equation (3.40)

$$W = 6\eta Q \int_{r_i}^{r_e} \frac{r}{H^3} \, dr \tag{3.42}$$

Similarly the equation for frictional torque can be written as

$$T = 2\pi\eta\Omega \int_{r_i}^{r_e} \frac{r^3}{H} \, dr \tag{3.43}$$

where Ω is the rotating angular velocity.

Now, for a rough hydrostatic bearing, the film-thickness is assumed to be approximated by

$$H \simeq h \left[1 + \sum_{j=1}^{M} \frac{\varepsilon_j}{h} \cos \frac{r}{a_j}\right] \tag{3.44}$$

where h is the nominal film-thickness, ε_j are the amplitudes and $2\pi\ a_j$ are the wave length of the roughness waves.

In the case of hydrodynamic lubrication $\frac{\varepsilon_j}{h} \ll 1$, and the function can be expanded and the various expressions mentioned before can be approximated as follows :

$$P_i = \frac{6\eta Q}{\pi h^3} \left[\ln \frac{1}{\lambda} - 3 \sum_{j=1}^{M} \frac{\varepsilon_j}{h} \int_{\lambda}^{1} \frac{\cos \lambda_j t}{t} dt + 3 \sum_{j=1}^{M} \frac{\varepsilon_j^2}{h^2} \left\{\ln \frac{1}{\lambda} + \int_{\lambda}^{1} \frac{\cos 2\lambda_j t}{t} dt\right\}\right] \tag{3.45}$$

$$W = \frac{6\eta Q\ r_e^2}{h^3} \left[\frac{1-\lambda^2}{2} - 3 \sum_{j=1}^{M} \frac{\varepsilon_j}{h} \int_{\lambda}^{1} t \cos \lambda_j t\, dt + 3 \sum_{j=1}^{M} \frac{\varepsilon_j^2}{h^2} \left\{\frac{1-\lambda^2}{2} + \int_{\lambda}^{1} t \cos 2 \lambda_j t\, dt\right\}\right] \tag{3.46}$$

$$T = \frac{2\pi\eta\Omega\ r_e^4}{h} \left[\frac{1-\lambda^4}{4} - \sum_{j=1}^{M} \frac{\varepsilon_j}{h} \int_{\lambda}^{1} t^3 \cos \lambda_j t\, dt + \frac{1}{2} \sum_{j=1}^{M} \frac{\varepsilon_j^2}{h^2} + \int_{\lambda}^{1} t^3 \cos 2 \lambda_j t\, dt\}\right] \tag{3.47}$$

where $r = r_e\ t$, $\lambda = \frac{r_i}{r_e}$, $\lambda_j = \frac{r_e}{a_j}$.

For fairly rough surfaces, the wave lengths of the roughness waves are sufficiently small and a_j tends to zero. In such a case λ_j tends to infinity and all the integrals [see appendix III] in the previous expressions are zero. Hence the expressions for determining

the flow flux, load capacity and frictional torque are given respectively as follows :

$$P_i = \frac{6\eta Q}{\pi h^3} \ln\frac{1}{\lambda} \{1 + 3 \sum_{j=1}^{M} \frac{\varepsilon_j^2}{h^2}\} \tag{3.48}$$

$$W = \frac{3\eta Q\, r_e^2}{h^3} (1 - \lambda^2) [1 + 3 \sum_{j=1}^{M} \frac{\varepsilon_j^2}{h^2}] \tag{3.49}$$

$$T = \frac{\pi\eta\Omega r_e^4}{2h} (1 - \lambda^4) [1 + \frac{1}{2} \sum_{j=1}^{M} \frac{\varepsilon_j^2}{h^2}] \tag{3.50}$$

It can be remarked here that, as the amplitudes of the roughness waves (roughness parameter) increase , the load capacity and frictional torque increase for a given flow flux. However, the flow flux decreases as σ increases for a given inlet pressure.

APPENDIX - III : EVALUATION OF INTEGRALS

It can be shown that the following integrals tends to zero as λ_j tends to infinity

(i) $\int_\lambda^1 \frac{\cos \lambda_j t}{t} dt$

$$= \frac{\sin \lambda_j}{\lambda_j} - \frac{\sin \lambda\lambda_j}{\lambda\lambda_j} + \int_\lambda^1 \frac{\cos \lambda_j t}{\lambda_j} \frac{dt}{t^2} \tag{3.51}$$

Since $\sin \lambda_j$ and $\cos \lambda_j t$ vary between -1 and +1 for all values of λ_j and are therefore bounded and hence

$$\frac{\sin \lambda_j}{\lambda_j} \to 0, \frac{\cos \lambda\lambda_j t}{\lambda_j} \to 0, \text{ as } \lambda_j \to \infty ,$$

and the above integral tends to zero as λ_j tends to ∞ .

(ii) Since

$$\int_{\lambda}^{1} t \cos \lambda_j t \, dt = \frac{\sin \lambda_j}{\lambda_j} + \frac{\cos \lambda_j}{\lambda_j} - \lambda \frac{\sin \lambda_j}{\lambda_j} - \frac{\cos \lambda_j}{\lambda_j^2} \tag{3.52}$$

and therefore by above argument this tends to zero as λ_j tends to ∞.

(iii) Similarly $\int_{\lambda}^{1} t^3 \cos \lambda_j t \, dt$ also tends to zero as λ_j tends to ∞.

APPENDIX-IV : DETERMINATION OF A_i's AND B_j's

The coefficients A_i's and B_j's in the equation (3.4-a) and (3.4-b) are so chosen that the mean square errors

$$E\{[Q_x - \frac{U}{2} \sum_{i=0}^{M} A_i H^i]^2\} \text{ and } E\{[Q_y - \frac{V}{2} \sum_{j=0}^{N} b_j H^j]^2\} \quad \text{are minimum,}$$

the conditions for which are as follows :

$$E[(Q_x H^{\nu}) - \frac{U}{2} \sum_{i=0}^{M} A_i \, E(H^{i+\nu})] = 0, \quad (\nu = 0,1,\ldots M) \tag{3.53-a}$$

$$E[(Q_y H^{\gamma}) - \frac{V}{2} \sum_{j=0}^{N} B_j \, E(H^{j+\gamma})] = 0, \quad (\gamma = 0,1,\ldots,N) \tag{3.53-b}$$

The equation (3.53-a) gives a system of (M+1) equations with (M+1) unknown A_i's provided $E(Q_x H^{\nu})$ and $E(H^{i+\nu})$ are known. By considering the density distribution function of h_s to be normal with variance σ, one can determine the values of $E(H^m)$ for any m [see appendix I]. The values of $E(Q_x H^{\nu})$ can be determined only when the correlation functions between Q_x and H^{ν} are known for all ν.

Since these correlation functions are not known we can determine them approximately for a given bearing system as follows. For a given bearing surface, the roughness profile can be traced by a Talysurf and thus, for each point of the bearing surface, a corresponding film-thickness H can be determined. Then some representative values of H say $H_1, H_2, \ldots, H_{N_o}$ are chosen. The fluxes $Q_1, Q_2, \ldots, Q_{N_o}$ corresponding to each of these representative values of film-thickness can be determined from the cases of bearings with smooth surfaces having film-thicknesses $H_1, H_2, \ldots, H_{N_o}$. By assuming that a given rough bearing is equivalent to number of tiny bearings with smooth surfaces joined together, we can write $E(Q_x H^\nu)$ approximately as follows :

$$E(Q_x H^\nu) = \frac{1}{N_o} \sum_{i=1}^{N_o} Q_{x_i} H_i^\nu \quad , \; (\nu = 0, 1, \ldots, M) \qquad (3.54)$$

Similarly, the values $E(Q_y H^\gamma)$ for $(\gamma = 0, 1, \ldots, N)$ can also be found. Thus, when $E(H^\nu)$, $E(Q_x H^\nu)$ and $E(Q_y H^\gamma)$ are known, all the coefficients A_i's and B_j's can be determined approximately.

LIST OF NOTATIONS

b	width of the bearing
$E(\ )$	expected value or statistical mean
F	frictional force
h, h_α	nominal film thickness
h_s	stochastic film thickness
H, H_α	total film thickness
kL	step position
L	Length of the step bearing
M, N, m, n	real numbers
P, P_i	pressure
Q, Q_i	flow flux
r	radial coordinate
r_i	inlet radious
T	torgue
U, V	surface velocity components in x and y-direction
W	load capacity
x, y, z	coordinate system
α	suffix
η	lubricant viscosity (constant)
σ^2	variance
μ	friction coefficient
ξ	random variable

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CHAPTER - IV

EFFECTS OF VISCOSITY VARIATION IN STOCHASTIC LUBRICATION

4.1 INTRODUCTION

In the previous Chapter, a stochastic form of Reynolds equation applicable to rough bearings has been derived by considering the flow of an incompressible lubricant, having a constant viscosity. As described in Chapter - II, viscosity can vary with temperature and pressure etc., it is desirable, therefore, to consider the variation of lubricant viscosity across as well as along the fluid film while studying the behaviour of lubricated systems. One of the approaches to study the effects of viscosity variation was suggested by Tipei [1962] by considering a typical viscosity film-thickness relation. Tipei and Nica [1967], Tipei and Degueurce [1974] have studied the effects of viscosity variation in the cases of thermohydrodynamic problems and shown that the solutions obtained for viscosity-film thickness relation, considered for plane or curved surfaces with convergent film, are quite satisfactory. Performance of viscosity variation across the fluid film with various profiles have also been studied by Quale and Wiltshire [1972]. It was pointed out that the coefficient of friction reduced significantly if the viscosity of the lubricant is made to vary across the thickness of the film.

Keeping in view of these, in this Chapter, a generalized form of Reynolds equation for stochastic lubrication is derived by considering viscosity-film thickness relation as suggested by Tipei [1962]. In particular, the case of one-dimensional rough slider bearing has been studied.

Further, a generalized form of Reynolds equation for stochastic lubrication for multiple layers of lubricant having different viscosities between two rough surfaces is derived by assuming the corresponding Reynolds equation as derived in Chapter II, without slip. The corresponding cases for longitudinal and transverse, one-dimensional roughness are given. The case of rough hydrostatic bearing with three layers of lubricant with different viscosities has been studied as a particular case.

4.2 VISCOSITY VARIATION ALONG THE FILM

Consider the flow of an incompressible lubricant having variable viscosity between a stationary rough surface and a moving smooth surface [see Fig.(4.1)]. Following Christensen [1969-70] it is assumed that the equation governing the pressure in the thin film is given by

$$\frac{\partial}{\partial x}\left(\frac{H^3}{\eta}\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{H^3}{\eta}\frac{\partial P}{\partial y}\right) = 6U\frac{\partial H}{\partial x} + 6V\frac{\partial H}{\partial y} \tag{4.1}$$

The following viscosity film thickness relation is considered, Tipei [1962], Tipei and Degueurce [1974] ,

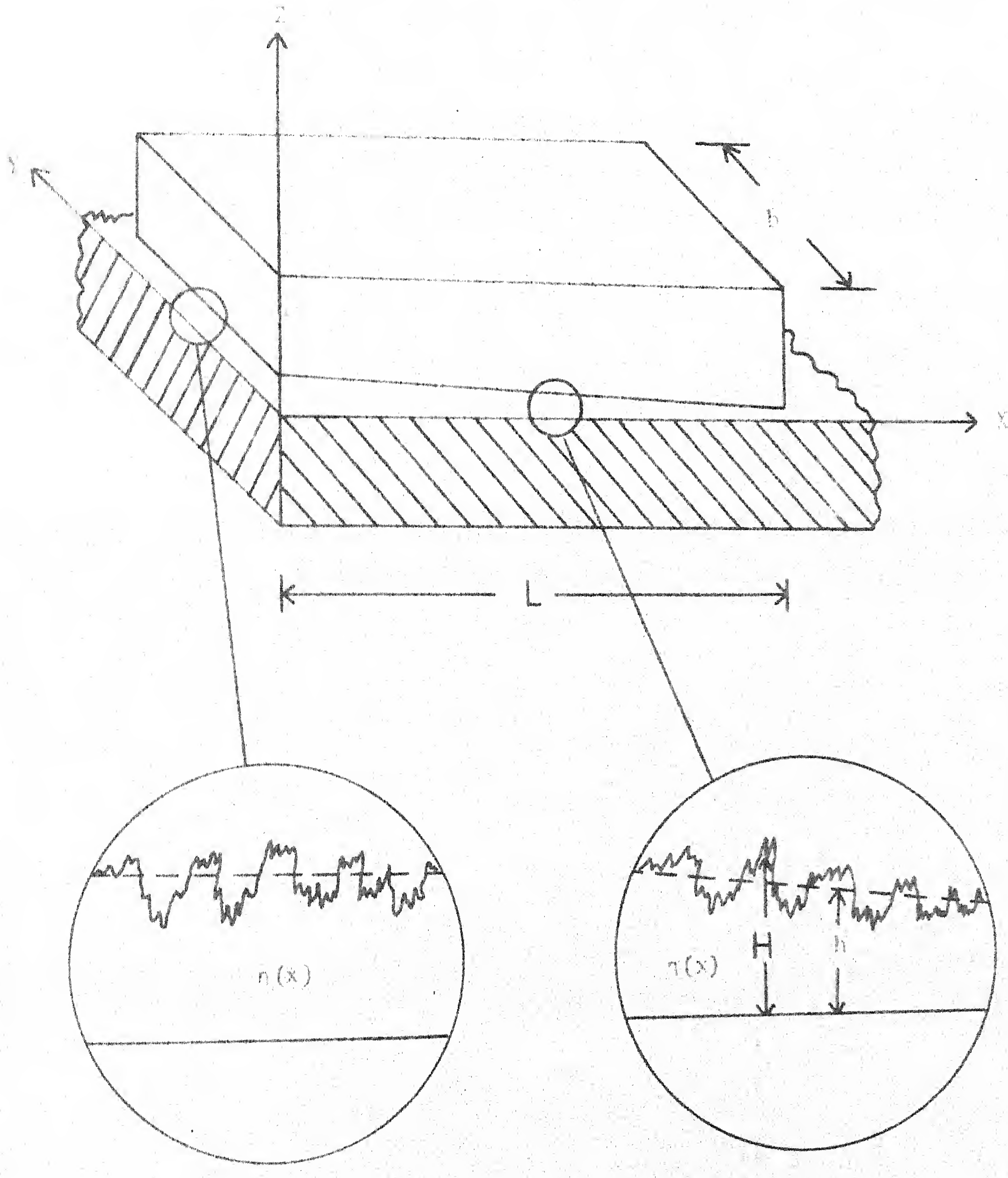

Fig. 4.1 TOPOGRAPHY OF SURFACES UNDER LUBRICATION WITH VISCOSITY VARIATION

$$\frac{\eta}{\eta_o} = \left(\frac{H}{h_1}\right)^q \quad , \qquad 0 \leq q \leq 1 \tag{4.2}$$

where the value of 'q' depends on the operating conditions of the bearing. Since $\frac{H}{h_1} \leq 1$, it can be noted that the viscosity decreases along the film-thickness and it is highest when $q = 0$ and lowest when $q = 1$.

It is observed that the pressure in the fluid film is a stochastic process as the film-thickness function H is a stochastic process and can be determined by averaging process, Papoulis [1965] and Peklenik [1968] . Proceeding as in the previous Chapter – III and using equation (4.1) and (4.2), we have

$$\frac{\partial}{\partial x} E\left[h_1^q H^{3-q} \frac{\partial P}{\partial x}\right] + \frac{\partial}{\partial y} E\left[h_1^q H^{3-q} \frac{\partial P}{\partial y}\right]$$

$$= 6\eta_o U \frac{\partial}{\partial x} E(H) + 6\eta_o V \frac{\partial}{\partial y} E(H) \tag{4.3}$$

where,

$$E(s) = \int_{-\infty}^{\infty} s \, F(s) \, ds.$$

and F(s) is the probability density distribution function of the stochastic variable s. The film-thickness function H can be written as

$$H = h(x,y) + h_s(x,y,\xi) \tag{4.4}$$

as described in Chapter – III.

As before, it is assumed that the flow fluxes which are stochastic process can be written in powers of H, Sveshnikov [1966] ,

Lin [1967], Christensen, et. al. [1975].

$$Q_x = \frac{U}{2} H - \frac{H^3}{12\eta} \frac{\partial P}{\partial x} = \frac{U}{2} H - \frac{h_1^q H^{3-q}}{12\eta_o} \frac{\partial P}{\partial x} = \frac{U}{2} \sum_{i=0}^{M} A_i H^i \qquad (4.5\text{-a})$$

$$Q_y = \frac{V}{2} H - \frac{H^3}{12\eta} \frac{\partial P}{\partial y} = \frac{U}{2} H - \frac{h_1^q H^{3-q}}{12\eta_o} \frac{\partial P}{\partial y} = \frac{V}{2} \sum_{j=0}^{N} B_j H^j \qquad (4.5\text{-b})$$

where A_i's and B_j's are non-random functions of x,y,U and V or constants depending upon the type of roughness, roughness slopes, minimum film-thickness etc. and may be determined theoretically or experimentally [see Chapter III].

Following the same procedure for an analytical evaluation as discussed in the Chapter - III and using equations (4.3), (4.4) and (4.5), the equation governing the fluid pressure can be obtained as follows :

$$\frac{\partial}{\partial x}\left[\frac{E(H^m)\, h_1^q}{E(H^{m-3+q})} \frac{\partial}{\partial x} E(P)\right] + \frac{\partial}{\partial y}\left[\frac{E(H^n)\, h_1^q}{E(H^{n-3+q})} \frac{\partial}{\partial y} E(P)\right]$$

$$= 6\eta_o\, U \frac{\partial}{\partial x}\left[\sum_{i=0, i\neq m}^{M} A_i E(H^i) + \frac{E(H^m)}{E(H^{m-3+q})}\left\{E\left(\frac{1}{H^{2-q}}\right) - \sum_{i=0, i\neq m}^{M} A_i E(H^{i-3+q})\right\}\right]$$

$$+ 6\eta_o\, V \frac{\partial}{\partial y}\left[\sum_{j=0, j\neq n}^{N} B_j E(H^j) + \frac{E(H^n)}{E(H^{n-3+q})}\left\{E\left(\frac{1}{H^{2-q}}\right) - \sum_{j=0, j\neq n}^{N} B_j E(H^{j-3+q})\right\}\right]$$

$$(4.6)$$

Equation (4.6) represents the generalized form of Reynolds equation for stochastic lubrication applicable to rough bearings with viscosity variation. When q = 0, this equation (4.6) reduces to the same form as discussed in Chapter - III [see equation (3.8)].

In the following, the case of an infinite slider bearing with transverse, one-dimensional roughness is considered.

4.3 INFINITE SLIDER BEARING

Consider the case of an infinite slider bearing with transverse, one-dimensional roughness as shown in Fig. (4.2). Taking $A_{i \neq m} = 0$ for all i and m = 0 in the generalized equation (4.6), the Reynolds equation applicable to this case can be written as :

$$\frac{d}{dx}\left[\frac{h_1^q}{E(\frac{1}{H^{3-q}})}\frac{d}{dx}E(P)\right] = 6\eta_o U \frac{d}{dx}\left[\frac{E(\frac{1}{H^{2-q}})}{E(\frac{1}{H^{3-q}})}\right] \qquad (4.7)$$

Integrating equation (4.7) with respect to x, we have

$$\frac{d}{dx}E(P) = \frac{6\eta_o U}{h_1^q}\left[E(\frac{1}{H^{2-q}}) - CE(\frac{1}{H^{3-q}})\right] \qquad (4.8)$$

where C is an integration constant.

Integrating equation (4.8) further and using boundary conditions

$$\begin{aligned} E(P) &= 0 \quad \text{at} \quad x = 0 , \\ E(P) &= 0 \quad \text{at} \quad x = L \end{aligned} \qquad (4.9)$$

we get,

$$E(P) = \frac{6\eta_o U}{h_1^q}\int_0^x \left[E(\frac{1}{H^{2-q}}) - CE(\frac{1}{H^{3-q}})\right] dx \qquad (4.10)$$

where

$$C = \frac{\int_0^L E(\frac{1}{H^{2-q}})\, dx}{\int_0^L E(\frac{1}{H^{3-q}})\, dx} \qquad (4.11)$$

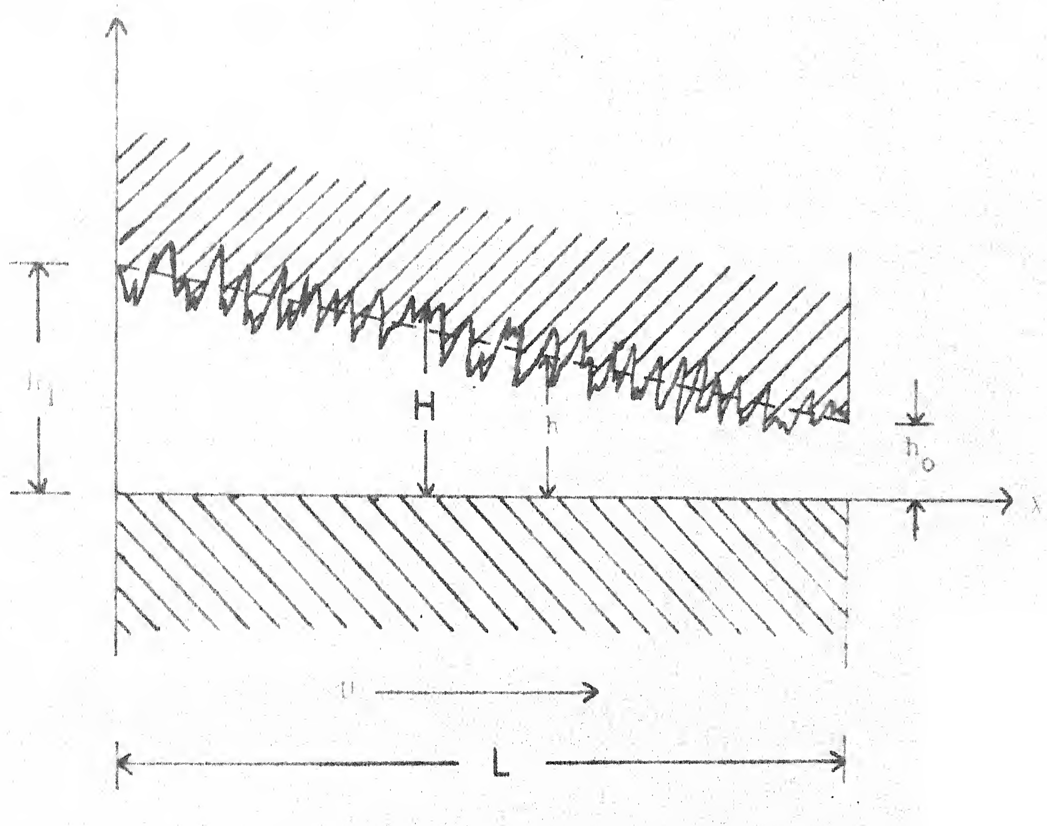

Fig. 4.2 ROUGH SLIDER BEARING

The expected value of load capacity is obtained by integrating the expression of E(P) over the bearing surface area as follows :

$$E(W) = \int_0^L E(P)\,dx = -\int_0^L x\,\frac{d}{dx}[E(P)]\,dx \qquad (4.12)$$

This gives,

$$E(W) = -\frac{6\eta_o U}{h_1^q}\int_0^L x\,[E(\frac{1}{H^{2-q}}) - CE(\frac{1}{H^{3-q}})]\,dx \qquad (4.13)$$

The expected value of shear stress on the runner is

$$E(\tau) = E([-\eta\frac{\partial u}{\partial z}]_{z=0}) = \frac{\eta_o U}{h_1^q}\,[4E(\frac{1}{H^{1-q}}) - 3CE(\frac{1}{H^{2-q}})] \qquad (4.14)$$

The friction force over runner is obtained by integrating the above expression of $E(\tau)$ as follows :

$$E(F) = \frac{\eta_o U}{h_1^q}\int_0^L [4E(\frac{1}{H^{1-q}}) - 3CE(\frac{1}{H^{2-q}})]\,dx \qquad (4.15)$$

By using approximate values of various quantities involved in the above equations (see Appendix), the expressions for E(W), E(F) and $E(\mu)$ can be calculated approximately as follows for $0 \leq q \leq 1$.

For $0 < q < 1$,

$$\overline{E(W)} = \frac{E(W)\,m^2}{6\eta_o U} = [\{\frac{1}{q} + \frac{1}{(1-q)}\frac{1}{\bar{h}_1} + \frac{1}{q(1-q)}\frac{1}{\bar{h}_1^q}\}$$

$$- C_q\{\frac{1}{(1-q)(2-q)}\frac{1}{\bar{h}_1^q} + \frac{1}{(2-q)}\frac{1}{\bar{h}_1^2} + \frac{1}{(1-q)}\frac{1}{\bar{h}_1}\}]$$

$$+ \frac{1}{2}[\{\frac{(2-q)}{\bar{h}_1^3} - \frac{(3-q)}{\bar{h}_1^2} + \frac{1}{\bar{h}_1^q}\}$$

$$- C_q\{\frac{(3-q)}{\bar{h}_1^4} - \frac{(4-q)}{\bar{h}_1^3} + \frac{1}{\bar{h}_1^q}\}]\,(\frac{\sigma}{h_o})^2 + \sigma^4(0) \qquad (4.16)$$

$$\overline{E(F)} = \frac{E(F)m}{\eta_o U} = \Big[\frac{4}{q}\Big(1 - \frac{1}{\bar{h}_1^q} - \frac{3\,C_q}{(1-q)}\Big(\frac{1}{\bar{h}_1^q} - \frac{1}{\bar{h}_1}\Big)\Big]$$

$$+ \Big[2(1-q)\Big(\frac{1}{\bar{h}_1^q} - \frac{1}{\bar{h}_1^2}\Big) - \frac{3}{2}C_q(2-q)\Big(\frac{1}{\bar{h}_1^q} - \frac{1}{\bar{h}_1^3}\Big)\Big]\Big(\frac{\sigma}{h_o}\Big)^2 + \sigma^4(0)$$

(4.17)

where,

$$C_q = \frac{(2-q)\Big(\frac{1}{\bar{h}_1^q} - \frac{1}{\bar{h}_1}\Big) + \frac{(2-q)^2}{2}\Big(\frac{1}{\bar{h}_1^q} - \frac{1}{\bar{h}_1^3}\Big) + \sigma^4(0)}{(1-q)\Big(\frac{1}{\bar{h}_1^q} - \frac{1}{\bar{h}_1^2}\Big) + \frac{(1-q)(3-q)}{2}\Big(\frac{1}{\bar{h}_1^q} - \frac{1}{\bar{h}_1^4}\Big) + \sigma^4(0)} \tag{4.18}$$

For $q = 0$,

$$\overline{E(W)} = \frac{E(W)\,m^2}{6\eta_o U} = \Big[\ln \bar{h}_1 - \Big(1 - \frac{1}{\bar{h}_1}\Big) - \frac{C_o}{2}\Big(1 - \frac{1}{\bar{h}_1}\Big)^2\Big]$$

$$+ \Big[\Big(\frac{1}{2} - \frac{3}{2}\frac{1}{\bar{h}_1^2} + \frac{1}{\bar{h}_1^3}\Big) - \frac{C_o}{2}\Big(1 - \frac{4}{\bar{h}_1^3} + \frac{3}{\bar{h}_1^4}\Big)\Big]\Big(\frac{\sigma}{h_o}\Big)^2 + \sigma^4(0)$$

(4.19)

$$\overline{E(F)} = \frac{E(F)\,m}{\eta_o U} = 4\ln \bar{h}_1 - 3C_1\Big(1 - \frac{1}{\bar{h}_1}\Big)$$

$$+ \quad 2\Big(1 - \frac{1}{\bar{h}_1^2}\Big) - 3C_o\Big(1 - \frac{1}{\bar{h}_1^3}\Big)\quad\Big(\frac{\sigma}{h_o}\Big)^2 + \sigma^4(0) \tag{4.20}$$

where,

$$C_o = 2\,\frac{\Big(1 - \frac{1}{\bar{h}_1}\Big) + \Big(1 - \frac{1}{\bar{h}_1^3}\Big)\Big(\frac{\sigma}{h_o}\Big)^2 + \sigma^4(0)}{\Big(1 - \frac{1}{\bar{h}_1^2}\Big) + 3\Big(1 - \frac{1}{\bar{h}_1^4}\Big)\Big(\frac{\sigma}{h_o}\Big)^2 + \sigma^4(0)} \tag{4.21}$$

For $q = 1$,

$$\overline{E(W)} = \frac{E(W)\,m^2}{6\eta_o U} = \Big[\Big(1 - \frac{1}{\bar{h}_1} - \frac{\ln \bar{h}_1}{\bar{h}_1}\Big) + C_1\Big(\frac{1}{\bar{h}_1} - \frac{\ln \bar{h}_1}{\bar{h}_1}\Big)\Big]$$

$$+ \frac{1}{2\bar{h}_1}\Big[\Big(1 - \frac{1}{\bar{h}_1}\Big)^2 - C_1\Big(1 - \frac{3}{\bar{h}_1^2} + \frac{2}{\bar{h}_1^3}\Big)\Big]\Big(\frac{\sigma}{h_o}\Big)^2 + \sigma^4(0) \tag{4.22}$$

$$\overline{E(F)} = \frac{E(F)\, m}{\eta_o\, U} = [4\,(1 - \frac{1}{\bar{h}_1}) - 3C_1 \frac{\ln \bar{h}_1}{\bar{h}_1}]$$

$$- \frac{3C_1}{2} (\frac{1}{\bar{h}_1} - \frac{1}{\bar{h}_1^3}) (\frac{\sigma}{h_o})^2 + \sigma^4(0) \qquad (4.23)$$

where,

$$C_1 = \frac{\dfrac{\ln \bar{h}_1}{\bar{h}_1} + \dfrac{1}{2} (\dfrac{1}{\bar{h}_1} - \dfrac{1}{\bar{h}_1^3}) (\dfrac{\sigma}{h_o})^2 + \sigma^4(0)}{(\dfrac{1}{\bar{h}_1} - \dfrac{1}{\bar{h}_1^2}) + (\dfrac{1}{\bar{h}_1} + \dfrac{1}{\bar{h}_1^4})(\dfrac{\sigma}{h_o})^2 + \sigma^4(0)} \qquad (4.24)$$

The coefficient of friction on the runner can be defined in the non-dimensional form as follows :

$$\overline{E(\mu)} = \frac{\overline{E(F)}}{\overline{E(W)}} \qquad (4.25)$$

To see the effects of roughness and viscosity variation the expressions for $\overline{E(W)}$, $\overline{E(F)}$ and $\overline{E(\mu)}$ are plotted in the Figs. (4.3), (4.4) and (4.5) for q = 0.0 , 0.5 and 1.0 respectively for different values of $\frac{\sigma}{h_o}$. It can be seen from Figs. (4.3), (4.4) and (4.5), that the load capacity and friction force increase for all values of q as $\frac{\sigma}{h_o}$ increases but the coefficient of friction decreases as $\frac{\sigma}{h_o}$ increases. Further, it can be seen from Figs. (4.3) and (4.4) that the case corresponding to q = 0.0 attains higher load capacity and more friction force than that of the cases corresponding to q = 0.5 and q = 1.0. However, from Fig. (4.5) it has been seen that the coefficient of friction in the case q = 0.0 increases upto a critical value of $\frac{\sigma}{h_o} \simeq 0.18$ and then decreases as $\frac{\sigma}{h_o}$ increases from this critical value.

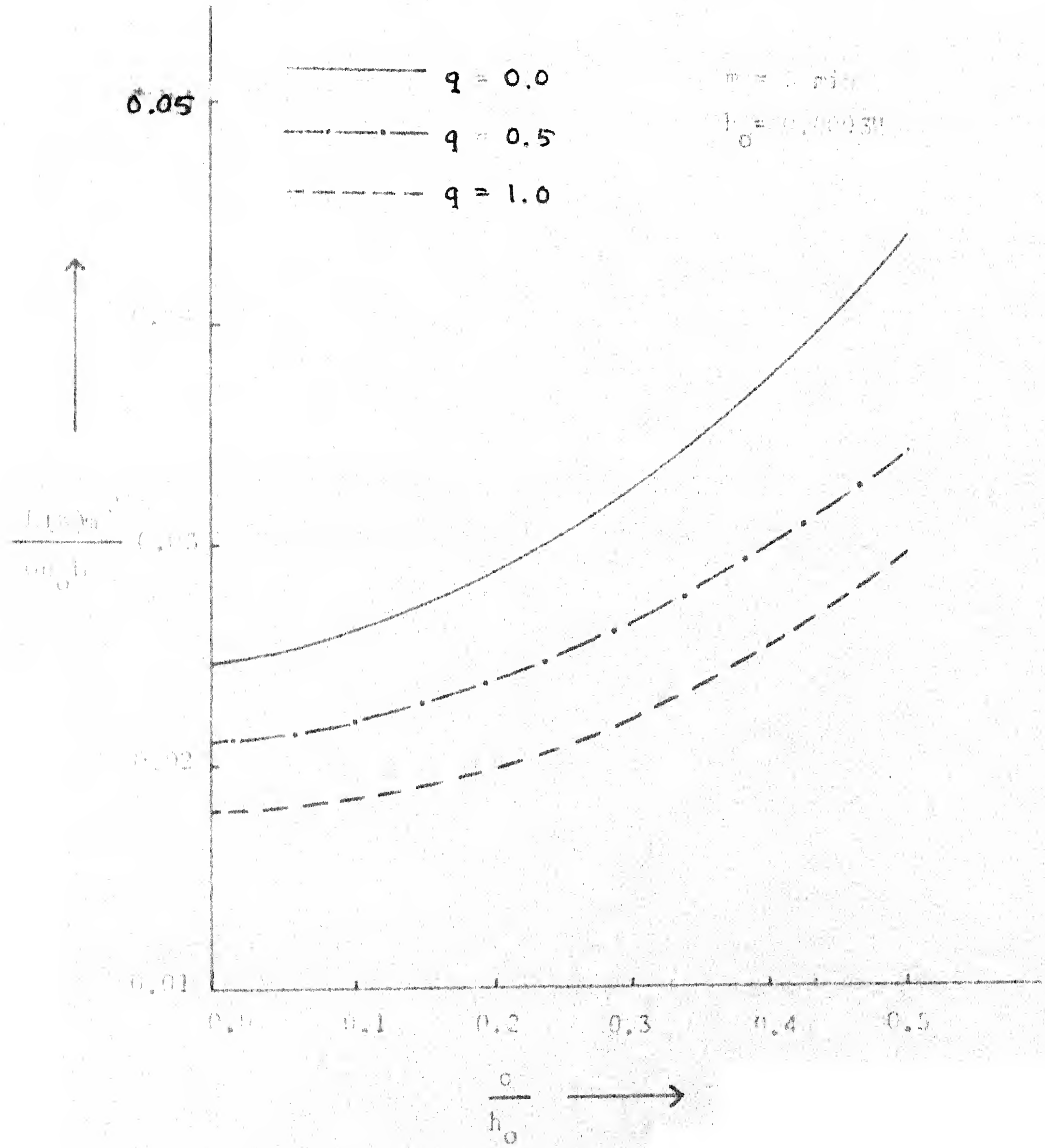

Fig. 4.3 LOAD CAPACITY

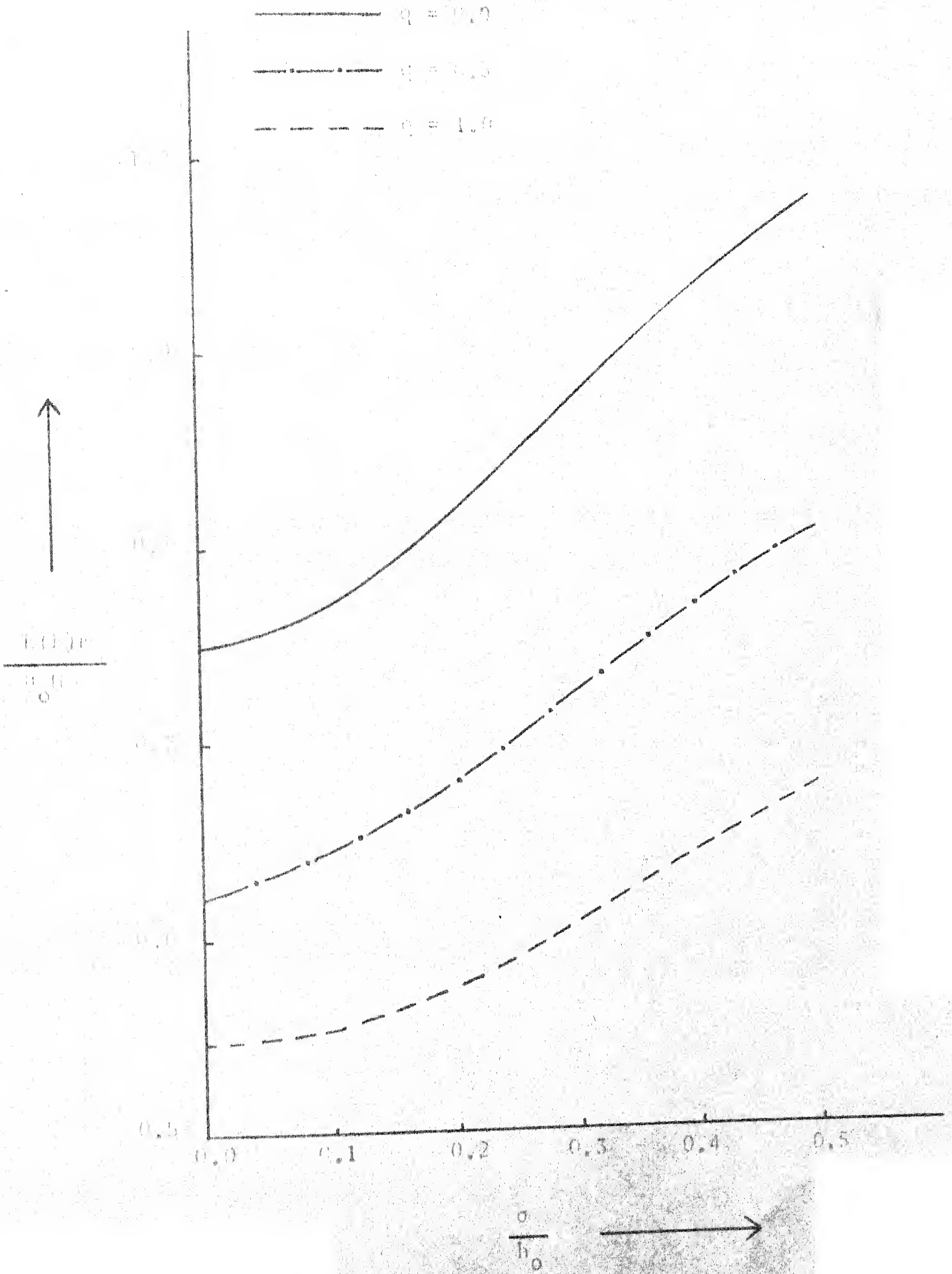

Fig. 4.4 FRICTION FORCE

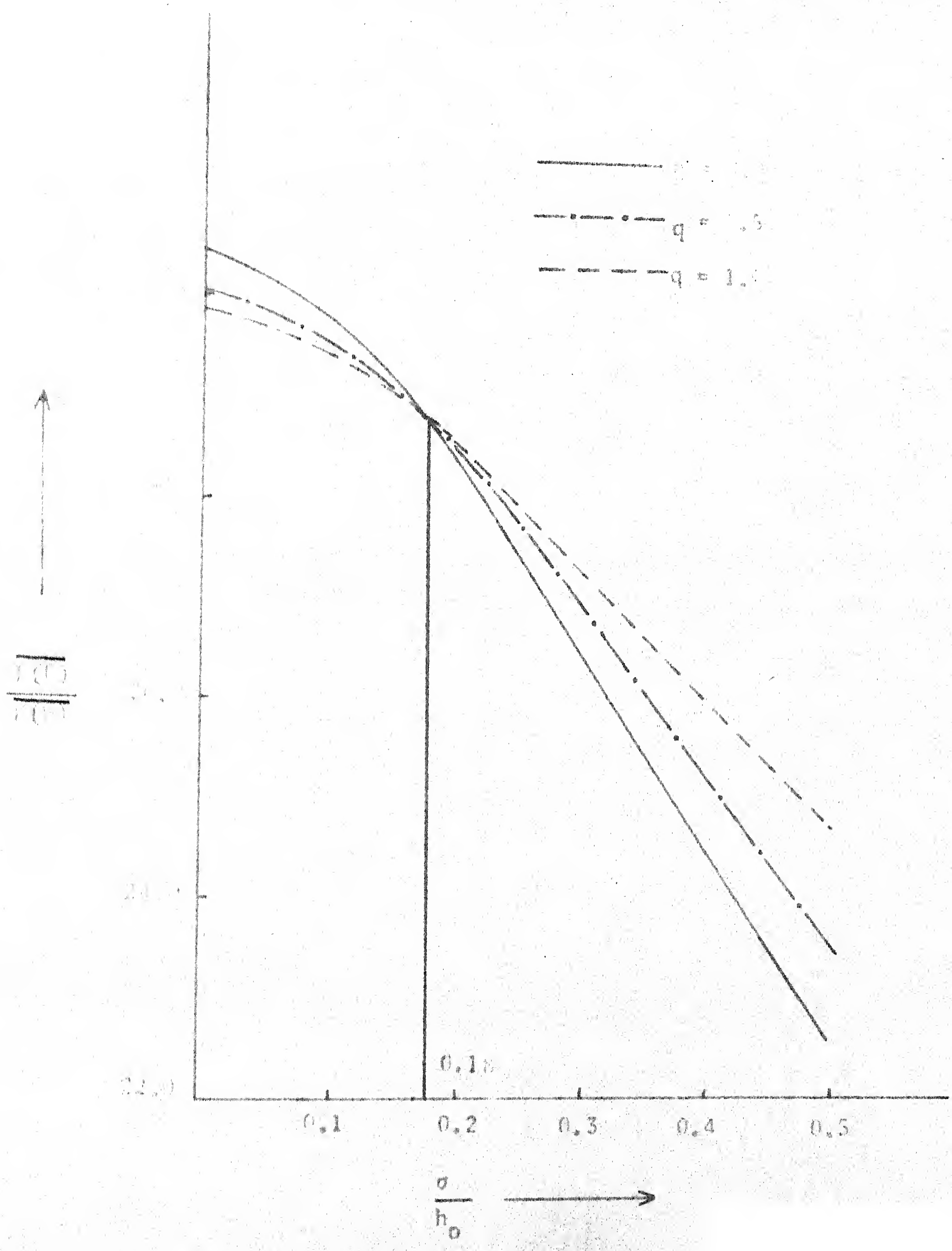

Fig. 4.5 COEFFICIENT OF FRICTION

4.4 VISCOSITY VARIATION ACROSS THE FILM

In the previous section 4.2, a generalized form of Reynolds equation applicable to rough bearings is derived by considering the variation of fluid viscosity along the film. In this section, we derive the stochastic Reynolds equation when viscosity varies across the film as a step function. As derived in Chapter - II the Reynolds equation in the two-dimensional form, without slip applicable to three layers of fluid can be rewritten as follows, [see equation (2.40) and Fig. (4.6)]

$$\frac{\partial}{\partial x}\left(F_1 \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(F_1 \frac{\partial P}{\partial y}\right) = \frac{U}{2}\frac{\partial G}{\partial x} + \frac{\partial}{\partial t}(h+H) \tag{4.26}$$

where

$$F_1 = \frac{h^3}{12\eta_2} + \frac{(h+H)^3 - h^3}{12\eta_1}$$
$$G = (h+H) \tag{4.27}$$

Fluxes in x and y direction are given by

$$Q_x = \frac{U}{2} G - \frac{F_1}{12}\frac{\partial P}{\partial x}$$
$$Q_y = -\frac{F_1}{12}\frac{\partial P}{\partial y} \tag{4.28}$$

Considering the bearing configuration as shown in Fig. (4.6), the film-thickness in this case can be written as follows :

$$H = h_o + h_s$$
$$h+H = h + h_o + h_s = h_n + h_s \tag{4.29}$$

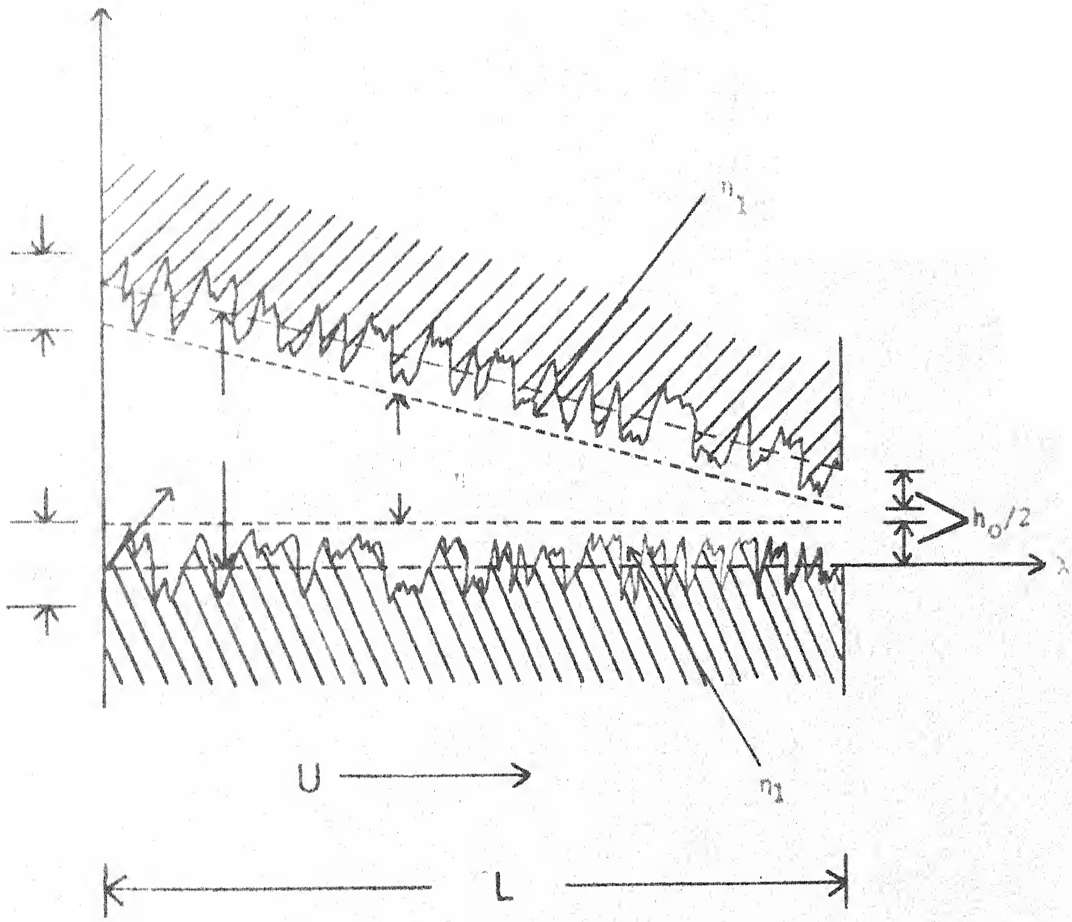

FIG. 4. ROUGH SLIDER BEARING WITH THREE LAYERS
OF LUBRICANT

where h_s is the stochastic part of the thickness and $\frac{h_o}{2}$ is the normal film thickness for upper and lower layers of lubricant having viscosity η_1.

Taking the expected value of equation (4.26), we have

$$\frac{\partial}{\partial x} E[F_1 \frac{\partial P}{\partial x}] + \frac{\partial}{\partial y} E [F_1 \frac{\partial P}{\partial y}] = \frac{U}{2} \frac{\partial}{\partial x} [E(G)] + \frac{\partial}{\partial t} [E(h_n + h_s)] \quad (4.30)$$

To evaluate the various terms in the above equation in the case of hydrodynamic lubrication, the ratio $\frac{h_s}{h_n}$ may be assumed to be small, i.e. $\frac{\sigma}{h_n} << 1$, we then have,

$$E(h_n + h_s) = h_n E(1 + \frac{h_s}{h_n}) = h_n + h_n E(\frac{h_s}{h_n}) \quad (4.31)$$

Assuming that the probability density distribution for h_s to be normal and having variance σ, the expected value of $E(h_n + h_s)$ can be written for $\frac{\sigma}{h_n} << 1$ as follows :

$$E(h_n + h_s) = h_n \quad (4.32)$$

By considering one-dimensional roughness pattern when the roughness direction is parallel or perpendicular to the sliding direction, as suggested by Christensen [1969-70] and Christensen and Tonder [1969] , the Reynolds equation in the following cases are derived from equation (4.30).

4.5 LONGITUDINAL, ONE-DIMENSIONAL ROUGHNESS

In this case, roughness is assumed to be of the form of long , narrow ridges and valleys running in the direction of sliding and the film thickness (h+H) is represented by,

where h_s is the stochastic part of the thickness and $\frac{h_o}{2}$ is the nominal film thickness for upper and lower layers of lubricant having viscosity η_1.

Taking the expected value of equation (4.26), we have

$$\frac{\partial}{\partial x} E[F_1 \frac{\partial P}{\partial x}] + \frac{\partial}{\partial y} E\, [F_1 \frac{\partial P}{\partial y}] = \frac{U}{2} \frac{\partial}{\partial x} \quad [E(G)] \; + \frac{\partial}{\partial t} \; [E(h_n + h_s)\,] \qquad (4.30)$$

To evaluate the various terms in the above equation in the case of hydrodynamic lubrication, the ratio $\frac{h_s}{h_n}$ may be assumed to be small, i.e. $\frac{\sigma}{h_n} << 1$, we then have,

$$E(h_n + h_s) = h_n \; E(1 + \frac{h_s}{h_n}) = h_n + h_n \; E(\frac{h_s}{h_n}) \qquad (4.31)$$

Assuming that the probability density distribution for h_s to be normal and having variance σ, the expected value of $E(h_n + h_s)$ can be written for $\frac{\sigma}{h_n} << 1$ as follows :

$$E(h_n + h_s) = h_n \qquad (4.32)$$

By considering one-dimensional roughness pattern when the roughness direction is parallel or perpendicular to the sliding direction, as suggested by Christensen [1969-70] and Christensen and Tonder [1969] , the Reynolds equation in the following cases are derived from equation (4.30).

4.5 LONGITUDINAL, ONE-DIMENSIONAL ROUGHNESS

In this case, roughness is assumed to be of the form of long , narrow ridges and valleys running in the direction of sliding and the film thickness (h+H) is represented by,

$$h+H = h_n(x,y,t) + h_s(y,\xi) \tag{4.33}$$

where h_n is nominal film thickness and h_s is the stochastic part of the film-thickness.

Following Christensen [1969-70] and assuming that $\frac{\partial P}{\partial x}$ is a variable with zero variance, $\frac{\partial P}{\partial x}$ and F_1 can be considered to be (approximately) stochastically independent quantities. Then we can write,

$$\frac{\partial}{\partial x} E(F_1 \frac{\partial P}{\partial x}) = E(F_1) \frac{\partial}{\partial x} [E(P)] \tag{4.34}$$

Also as flow flux in the y-direction do not vary randomly,

$$E [\frac{\partial}{\partial y} (F_1 \frac{\partial P}{\partial y})]$$

can be found as follows :

Considering

$$F_1 \frac{\partial P}{\partial y} = q_y(x,y) \tag{4.35}$$

which do not vary randomly, we write

$$E(\frac{\partial P}{\partial y}) = \frac{\partial}{\partial y} E(P) = q_y \; E(\frac{1}{F_1}) \tag{4.36}$$

and,

$$q_y = E(F_1 \frac{\partial P}{\partial y}) = \frac{1}{E(1/F_1)} \frac{\partial}{\partial y} E(P) \tag{4.37}$$

Putting the expressions (4.32), (4.34) and (4.37) into the equation (4.30), we get

$$\frac{\partial}{\partial x} [E(F_1) \frac{\partial}{\partial x} E(P)] + \frac{\partial}{\partial y} [\frac{1}{E(1/F_1)} E(P)] = \frac{U}{2} \frac{\partial}{\partial x} E(G) + \frac{\partial}{\partial t} (h_n) \tag{4.38}$$

This is the Reynolds-type equation for a bearing with three layers and applicable for longitudinal, one-dimensional roughness.

4.6 TRANSVERSE, ONE-DIMENSIONAL ROUGHNESS

In this case surface roughness is assumed to have the form of long, narrow ridges and deep valleys running in the y-direction (perpendicular to the sliding direction). Then, the film-thickness is given by

$$h + H = h_n(x,y,t) + h_s(x,\xi) \tag{4.39}$$

Following Christensen [1969-70] and carrying out a similar calculation as in the previous case, we get

$$\frac{\partial}{\partial x}\left[\frac{1}{E(1/F_1)}\frac{\partial}{\partial x}E(P)\right] + \frac{\partial}{\partial y}\left[E(F_1)\frac{\partial}{\partial y}E(P)\right] = \frac{\mu}{2}\frac{\partial}{\partial x}\left[\frac{E(G/F_1)}{E(1/F_1)}\right] + \frac{\partial}{\partial t}(h_n) \tag{4.40}$$

which is a Reynolds-type equation for a bearing with three layers of lubricant and applicable for transverse, one-dimensional roughness.

Similarly the stochastic equations governing the pressure in cases of other forms of roughness can be derived, Christensen [1969-70].

4.7 HYDROSTATIC BEARING WITH THREE LAYERS

Consider the flow of three layers of lubricant in a rough hydrostatic bearing as shown in Fig. (4.7). The Reynolds equation for longitudinal roughness in this case can be written as :

$$\frac{\partial}{\partial r}\left[r\, E(F_1)\frac{\partial}{\partial r}E(P)\right] = 0 \tag{4.41}$$

where F_1 is given by equation (4.27).

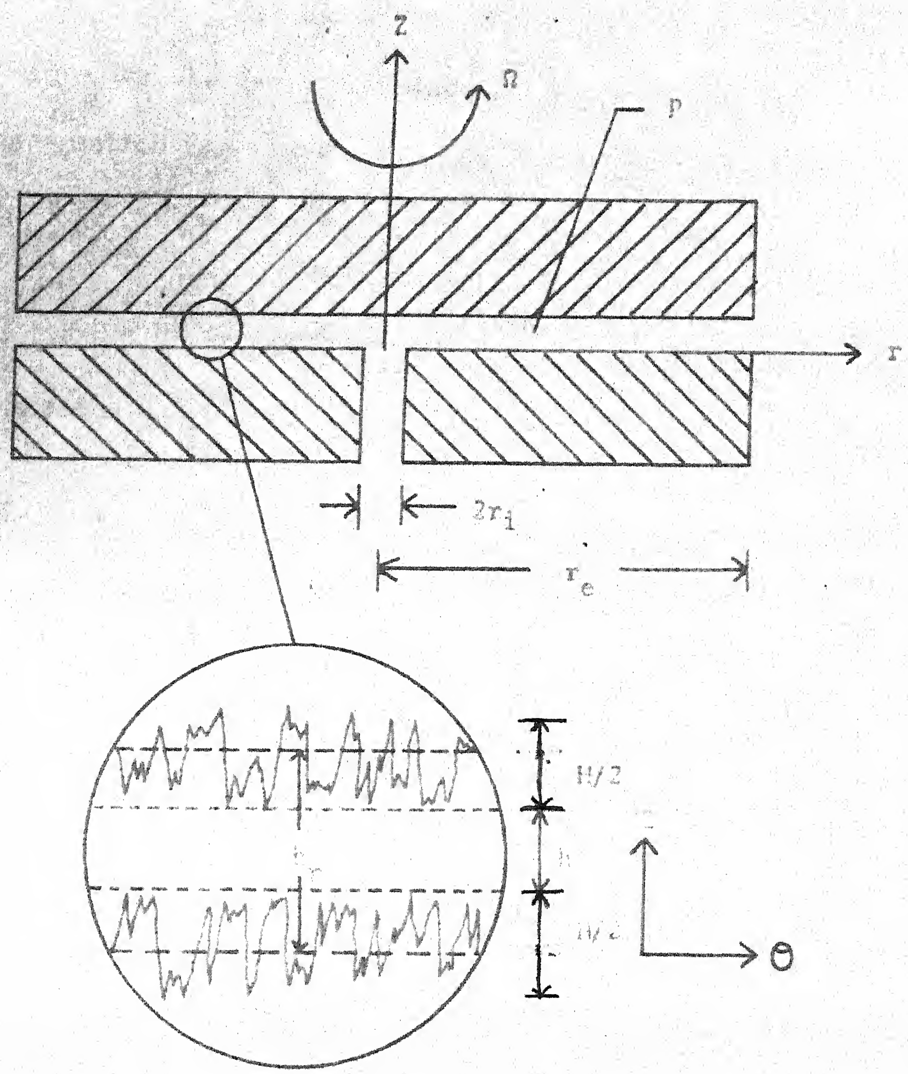

Fig. 4.7 LONGITUDINALLY ROUGH HYDROSTATIC BEARING

Integrating equation (4.41) and using the definition of flow flux, we get,

$$E(Q) = -2\pi r E(F_1) \frac{\partial}{\partial r} E(P) \tag{4.42}$$

The expressions for the pressure and flow flux can be obtained, by integrating equation (4.42) and using the usual boundary conditions $E(P) = E(P_i)$ at $r = r_i$, and $E(P) = 0$ at $r = r_e$, as follows :

$$E(P) = \frac{E(Q)}{2\pi E(F_1)} \ \ell n \frac{r_e}{r} \tag{4.43}$$

$$E(P_i) = \frac{E(Q)}{2\pi E(F_1)} \ \ell n \left(\frac{r_e}{r_i}\right) \tag{4.44}$$

The Expected value of the load capacity can be calculated as usual and we get,

$$E(W) = \frac{E(Q)}{4E(F_1)} (r_e^2 - r_i^2) \tag{4.45}$$

Considering equation (4.27), for hydrodynamic lubrication as considered before, $E(F_1)$ can be written as :

$$E(F_1) = \frac{h^3}{12\eta_2} + \frac{E[(h_n + h_s)^3] - h^3}{12\eta_1}$$

$$= \frac{h^3}{12\eta_2} + \frac{h_n^3 [1 + 3(\frac{\sigma}{h_n})^2] - h^3}{12\eta_1} \tag{4.46}$$

It can be seen from equation (4.46) that $E(F_1)$ decreases as η_1 and η_2 increase and $E(F_1)$ increases as $(\frac{\sigma}{h_n})$ increases. Hence for a constant flow flux, the expected value of load capacity increases as η_1 and η_2 increase and $E(W)$ decreases as $\frac{\sigma}{h_n}$ increases. The last result is the same as discussed in Chapter - III for this case when m = 3

4.8 RESULTS AND DISCUSSIONS

In this Chapter, a stochastic Reynolds equation applicable to rough slider bearing is derived by considering the variation of fluid viscosity along the film through viscosity-film thickness relation and a particular case of rough infinite slider bearing is discussed. It has been shown that the load capacity and friction force decrease as the index parameter 'q' increases from zero to one while the load capacity and friction force increase with the roughness. It has also be seen that the coefficient of friction decreases as q increases upto a critical value of $\frac{\sigma}{h_o} \approx 0.18$, and after that it increases as q increases.

Further, the stochastic equations applicable to the longitudinal and transverse, one-dimensional roughness are also derived by considering three different layers of lubricant with different viscosities. As a particular case, a rough hydrostatic bearing is considered. It is shown that the load capacity increases as the viscosities of lubricant layers increase.

APPENDIX

In the case of hydrodynamic lubrication, the ratio $\frac{h_s}{h}$ may be assumed to be small where $\frac{\sigma}{h}$ <<1 and one can write,

$$H^n = [h+h_s]^n \simeq h^n [1 + n (\frac{h_s}{h}) + \frac{n(n-1)}{2} (\frac{h_s}{h})^2 + \ldots] \qquad (4.47)$$

Assuming that the probabulity density distribution for the stochastic variable h_s to be normal and having variance σ , the expected value of equation (4.47) can be written for $\frac{\sigma}{h} << 1$ as follows :

$$E(H^n) \simeq h^n [1 + \frac{n(n-1)}{2} (\frac{\sigma}{h})^2 + \sigma^4(0)] \qquad (4.48)$$

Similarly, one can write

$$E(\frac{1}{H^n}) \simeq \frac{1}{h^n} [1 + \frac{n(n+1)}{2} (\frac{\sigma}{h})^2 + \sigma^4(0)] \qquad (4.49)$$

For a slider bearing as shown in Fig. (4.2),

$$h = h_1 - mx \qquad (4.50)$$

where m-the angle of inclination between two surfaces. We can have,

$$dh = -m\, dx \qquad (4.51)$$

Now, using equation (4.51), we can write

$$\int_0^x E(H^n)\, dx = \frac{1}{m} \int_h^{h_1} E(H^n)\, dh \qquad (4.52)$$

and

$$\int_0^L E(H^n)\, dx = \frac{1}{m} \int_{h_o}^{h_1} E(H^n)\, dh \qquad (4.53)$$

In a similar manner, other expression can also be written for various values of n and integration can be carried out in the same way as discussed in Chapter III.

LIST OF NOTATIONS

$E(\ )$	expected value or statistical mean
F	friction force
h, h_1, h_o, h_n	nominal film thickness
h_s	stochastic film thickness
H	total film thickness
L	length of bearing
M, N, m, n	real numbers
P	pressure in the film of the lubricant
P_i	inlet pressure
Q, Q_x, Q_y	flow flux
q	power index
r, θ, z	cylindrical polar coordinate
r_i	inlet radius
r_e	outer radius
T	torque
U, V	surface velocity components in x and y directions
W	load capacity
x, y, z	rectangular coordinate
η	variable viscosity
η_o, η_1, η_2	viscosity of the base oil
ξ	random variable
σ	varience

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CHAPTER - V

A NEW THEORY OF MIXED LUBRICATION

5.1 INTRODUCTION

In the previous two chapters, generalized forms of Reynolds equation for hydrodynamic lubrication applicable to rough bearings have been derived. In the case of rough surfaces the hydrodynamic conditions are maintained by using a high viscosity lubricant such that the film-thickness is greater than the heights of roughness asperities. If it were not the case, asperities penetration may occur and boundary lubrication can exist under extreme condition where bearing surfaces are in continuous contact at the tips of the asperities which support the load. The formation of oxide film and adsorbtion of the lubricant on the surfaces play an essential role in the lubrication mechanism (such as reduction of friction) but the viscosity of the lubricant which may be present in between the groves in a minute quantity does not take important part in the process, Bowden and Tabor [1950,1964] .

When the fluid film, which seperates the two surfaces in a lubricated system becomes thin enough such that the surface asperitie begin to interfere, a situation between hydrodynamic and boundary lubrications, known as mixed lubrication arises. In this case, load

is partly supported by fluid film and partly by asperity-contacts. The frictional force depends upon the interaction of the surface asperities with the fluid film and the total friction would arise partly due to boundary lubrication conditions associated with asperity-contacts and partly from hydrodynamic friction, Fuller [1954].

Lenning [1960] has pointed out that the transition from boundary to mixed lubrication occurs by increasing base oil viscosity, decreasing surface roughness and decreasing applied load. Dobry [1964] has also considered a mathematical model to study the velocity of transition from boundary to mixed lubrication and showed that the velocity of transition depends upon the surface roughness, lubricant viscosity, load and external pressure. A model for the contact of hydrodynamically lubricated machine components was also developed by Thomson and William [1972] from which asperity to asperity contact load could be obtained.

Christensen [1969] and Fowels [1971] have considered the asperity deformation together with hydrodynamic lubrication effect for elastohydrodynamically lubricated bearings. Christensen [1972] has also presented a mathematical model for mixed lubrication which arise from an interaction of the surface asperities with hydrodynamic fluid film between the sliding surfaces. It has been shown that in mixed lubrication regime, friction is mainly controlled by lubrication properties of liquid-solid interface where as load is entirely

controlled by the hydrodynamic properties of the bearing. Tsao and Tong [1975] presented a model for mixed lubrication by representing the surface asperities by short plasto-elastic cylinders with spherical tops where the heights of asperities are assumed to have a Gaussion distribution. Solution of Reynolds equation was combined with boundary lubrication theory or the micro-elastohydrodynamic theory to study the condition of mixed lubrication.

In this chapter, a new theory of mixed lubrication is presented by considering the surface asperities to be randomly distributed fins or cones which penetrate through the fluid film and partly may be in contact with the asperities of the opposite surface. These asperities are considered to form a net work of interacting zone corresponding to each surface, through which the lubricant flows as happens in the case of flow through channels with singular resistances or some what as in the case of a flow through porous matrix. Such situation may also arise when the additive present in the lubricant, such as long chain polar compounds (fatty acids), get adsorbed on the sliding surfaces with their free tail penetrating into the film, and thus forming a thin interacting zone, Askwith, et. al. [1966].

For the sake of generality, it is considered that these interacting net work zones are seperated by a thin lubricant film whose viscosity may be different from that of the viscosities of the lubricants present in the two interacting zones due to surfactants etc., Askwith, et. al. [1966], Fein and Kreuz [1966]. Thus, the

effective lubricated region is divided into two interacting surface net work zones and a middle zone through which the simultaneous layers of fluid with different viscosities flow. In these interacting net work zones, modified forms of Navier-Stoke's equations are assumed and which can also be thought of as generalized forms of Darcy's law applicable to porous matrix, Brinkman [1949] . However, in the seperating film, the usual hydrodynamic equations are considered. Keeping these points in view, a generalized form of Reynolds equation for mixed lubrication is derived and particular forms applicable to various cases are deduced.

A typical situation where this theory can be applied is the case of synovial joints under impact loading conditions. These joints consist of two poroelastic surfaces called 'articular cartilage', with a fluid lubricant film between them. This fluid, known as synovial fluid, contains hyaluronic acid molecules, which changes its viscosity and also acts as a boundary lubricant at cartilage surfaces under extreme conditions of operation, Dowson [1967], Dowson et. al. [1970] , Chandra [1975] , Shukla and Chandra [1975] , . The superficial tangential zones of the Cartilage and the region of asperities peneration in the synovial fluid film may be considered as the interacting surface net work zones while studying synovial joints by the help of the theory presented in this chapter, Mittal and Millington [1971] , Mow and Redler [1974] .

5.2 BASIC EQUATIONS

Consider the flow of three incompressible layers of lubricants in the two interacting zones and the middle separating layer which might exist atleast in the region of hydrodynamic lubrication. For example, in the slider bearing, mixed lubrication can exist at the trailing edge while at the leading edge hydrodynamic lubrication can prevail, Christensen [1972]. When the mixed lubrication condition exists everywhere in the bearing contact area, the flow in the separating middle layer can be ignored. In general the physical situation and the coordinate systems described here are shown in Fig. (5.1).

By using the usual assumptions of lubrication theory and considering the same pressure gradient in the three layers, a generalized form of Reynolds equation is derived as follows :

In the interacting region I. ($H_1 \leq z \leq H_1 + H_\ell$), fluid motion is considered to be governed by the following modified equations of motion, Brinkmann [1949].

$$\eta_1 \frac{\partial^2 u_1}{\partial z^2} - \frac{\partial p}{\partial x} - \frac{\eta_1}{\phi_x} u_1 = 0 \qquad (5.1\text{-a})$$

$$\eta_1 \frac{\partial^2 v_1}{\partial z^2} - \frac{\partial p}{\partial y} - \frac{\eta_1}{\phi_y} v_1 = 0 \qquad (5.1\text{-b})$$

where $\frac{1}{\phi_x}$ and $\frac{1}{\phi_y}$ measure the degree of resistance to the flow and can depend upon the characteristics of the net works which in tern may

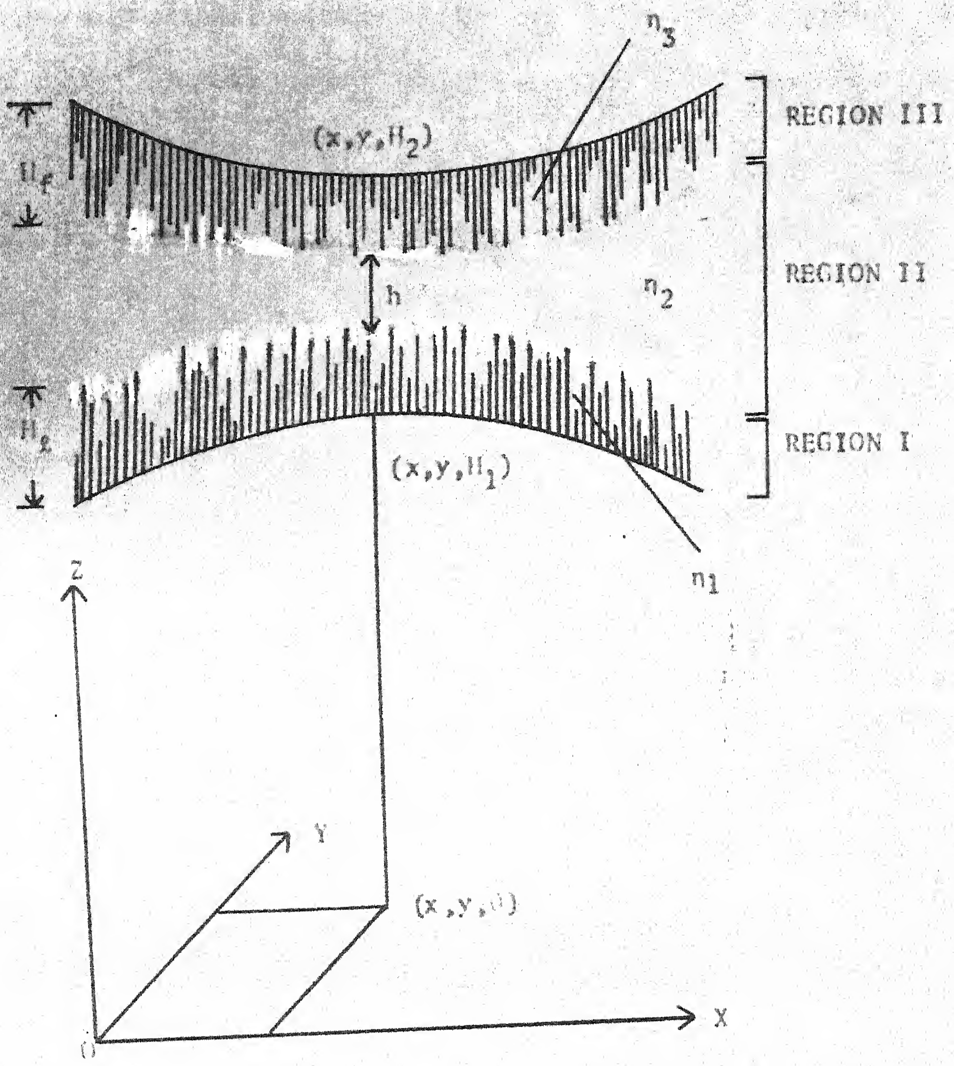

Fig. 5.1 COORDINATE SYSTEM

depend upon the type of asperity, its distribution, heights, shape, molecular structure, crystalography of adsorbed molecules etc. In the case of a synovial joint, these depend upon the characteristics of the superficial tangential zone of the cartilage surface and the molecular structure of the hyaluronic acid. In fact, when $\frac{1}{\phi_x}$, $\frac{1}{\phi_y}$ increase the residence to the flow increases and they are zero in the case of no resistance.

In the region II $(H_1 + H_\ell \leq z \leq H_1 + H_\ell + h)$, the fluid flow is governed by the usual equations of motion

$$\eta_2 \frac{\partial^2 u_2}{\partial z^2} - \frac{\partial p}{\partial x} = 0 \qquad (5.2\text{-a})$$

$$\eta_2 \frac{\partial^2 v_2}{\partial z^2} - \frac{\partial p}{\partial y} = 0 \qquad (5.2\text{-b})$$

The thickness of this hydrodynamic film layer may be negligible in that part of the bearing where mixed lubrication exists.

In the region III $(H_1 + H_\ell + h \leq z \leq H_2)$, similar forms of equations of motion as in region I are considered to study the flow characteristics in the second interacting zone.

$$\eta_3 \frac{\partial^2 u_3}{\partial z^2} - \frac{\partial p}{\partial x} - \frac{\eta_3}{\phi_x} u_3 = 0 \qquad (5.3\text{-a})$$

$$\eta_3 \frac{\partial^2 v_3}{\partial z^2} - \frac{\partial p}{\partial y} - \frac{\eta_3}{\phi_y} v_3 = 0 \qquad (5.3\text{-b})$$

The equation of continuity can be written as follows :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.4}$$

Now for the region I, integrating equations (5.1-a) and (5.1-b) with respect to z and using the following boundary conditions

$$\begin{aligned} u_1 &= U_1 \ , \ v_1 = V_1 \quad \text{at } z = H_1 \\ u_1 &= U_i \ , \ v_1 = V_i \quad \text{at } z = H_1 + H_\ell \end{aligned} \tag{5.5}$$

the expressions for fluid velocities can be written as :

$$u_1 = U_i \frac{\sinh \frac{z-H_1}{\sqrt{\phi_x}}}{\sinh \frac{H_\ell}{\sqrt{\phi_x}}} - U_1 \frac{\sinh \frac{z-(H_1+H_\ell)}{\sqrt{\phi_x}}}{\sinh \frac{H_\ell}{\sqrt{\phi_x}}} - \frac{\phi_x}{\eta_1}\left[1 - \frac{\sinh \frac{z-H_1}{\sqrt{\phi_x}} - \sinh \frac{z-(H_1+H_\ell)}{\sqrt{\phi_x}}}{\sinh \frac{H_\ell}{\sqrt{\phi_x}}}\right]\frac{\partial p}{\partial x} \tag{5.6-a}$$

and

$$v_1 = V_i \frac{\sinh \frac{z-H_1}{\sqrt{\phi_y}}}{\sinh \frac{H_\ell}{\sqrt{\phi_y}}} - V_1 \frac{\sinh \frac{z-(H_1+H_\ell)}{\sqrt{\phi_y}}}{\sinh \frac{H_\ell}{\sqrt{\phi_y}}} - \frac{\phi_y}{\eta_1}\left[1 - \frac{\sinh \frac{z-H_1}{\sqrt{\phi_y}} - \sinh \frac{z-(H_1+H_\ell)}{\sqrt{\phi_y}}}{\sinh \frac{H_\ell}{\sqrt{\phi_y}}}\right]\frac{\partial p}{\partial y} \tag{5.6-b}$$

In the region II, integration of equation (5.2-a) and (5.2-b) with the boundary conditions

$$u_2 = U_i, \; v_2 = V_i \text{ at } z = H_1 + H_\ell$$
$$u_2 = U_j, \; v_2 = V_j \text{ at } z = H_1 + H_\ell + h = H_2 - H_f \qquad (5.7)$$

give,

$$u_2 = \frac{1}{2\eta_2} [z^2 - z(2H_1 + 2H_\ell + h) + (H_1 + H_\ell)(H_1 + H_\ell + h)] \frac{\partial p}{\partial x}$$
$$+ \left(\frac{U_j - U_i}{h}\right) z + \frac{1}{h} [U_i(H_1 + H_\ell + h) - U_j(H_1 + H_\ell)] \qquad (5.8\text{-a})$$

and

$$v_2 = \frac{1}{2\eta_2} [z^2 - z(2H_1 + 2H_\ell + h) + (H_1 + H_\ell)(H_1 + H_\ell + h)] \frac{\partial p}{\partial y}$$
$$+ \left(\frac{V_j - V_i}{h}\right) z + \frac{1}{h} [V_i(H_1 + H_\ell + h) - V_j(H_1 + H_\ell)] \qquad (5.8\text{-b})$$

For the region III, again integrating equations (5.3-a) and (5.3-b) and using the boundary conditions

$$u_3 = U_j, \; v_3 = V_j \text{ at } z = H_1 + H_\ell + h = H_2 - H_f$$
$$u_3 = U_2, \; v_3 = V_2 \text{ at } z = H_2 \qquad (5.9)$$

the expressions for velocities can be written as

$$u_3 = U_j \frac{\sinh \dfrac{H_2 - z}{\sqrt{\phi_x}}}{\sinh \dfrac{H_f}{\sqrt{\phi_x}}} + U_2 \frac{\sinh \dfrac{z - (H_2 - H_f)}{\sqrt{\phi_x}}}{\sinh \dfrac{H_f}{\sqrt{\phi_x}}}$$
$$+ \frac{\phi_x}{\eta_3} \left[\frac{\sinh \dfrac{H_2 - z}{\sqrt{\phi_x}} + \sinh \dfrac{z - (H_2 - H_f)}{\sqrt{\phi_x}}}{\sinh \dfrac{H_f}{\sqrt{\phi_x}}} - 1 \right] \frac{\partial p}{\partial x} \qquad (5.10\text{-a})$$

$$v_3 = V_j \frac{\sinh \frac{H_2 - z}{\sqrt{\phi_y}}}{\sinh \frac{H_f}{\sqrt{\phi_y}}} + V_2 \frac{\sinh \frac{z - (H_2 - H_f)}{\sqrt{\phi_y}}}{\sinh \frac{H_f}{\sqrt{\phi_y}}}$$

$$+ \frac{\phi_y}{\eta_3} \left[\frac{\sinh \frac{H_2 - z}{\sqrt{\phi_y}} + \sinh \frac{z - (H_2 - H_f)}{\sqrt{\phi_y}}}{\sinh \frac{H_f}{\sqrt{\phi_y}}} - 1 \right] \frac{\partial p}{\partial y} \tag{5.10-b}$$

To find the expressions of U_i, V_i, U_j and V_j, shear stresses at the interface i at $z = H_1 + H_\ell$, and interface j at $z = H_1 + H_\ell + h = H_2 - H_f$ are equated as follow :

$$\left.\begin{aligned} \eta_1 \frac{\partial u_1}{\partial z} &= \eta_2 \frac{\partial u_2}{\partial z} \\ \eta_1 \frac{\partial v_1}{\partial z} &= \eta_2 \frac{\partial v_2}{\partial z} \end{aligned}\right\} \text{at } z = H_1 + H_\ell$$

$$\left.\begin{aligned} \eta_2 \frac{\partial u_2}{\partial z} &= \eta_3 \frac{\partial u_3}{\partial z} \\ \eta_2 \frac{\partial v_2}{\partial z} &= \eta_3 \frac{\partial v_3}{\partial z} \end{aligned}\right\} \text{at } z = H_2 - H_f \tag{5.11}$$

After using the expressions for velocity gradients for the three regions in equation (5.11), we have

$$U_i f_1 - U_j \frac{\eta_2}{h} = U_1 f_3 - g_1 \frac{\partial p}{\partial x} \tag{5.12-a}$$

$$-U_i \frac{\eta_2}{h} + U_j f_2 = U_2 f_4 - g_2 \frac{\partial p}{\partial x} \tag{5.12-b}$$

$$V_i f_1' - V_j \frac{\eta_2}{h} = V_1 \; f_3' - g_1' \frac{\partial p}{\partial y} \tag{5.12-c}$$

$$-V_i \frac{\eta_2}{h} + V_j \; f_2' = V_2 \; f_4' - g_2' \; \frac{\partial p}{\partial y} \tag{5.12-d}$$

where

$$f_1 = \frac{\eta_2}{h} + \frac{\eta_1}{\sqrt{\phi_x}} \; \frac{1}{\tanh \frac{H_\ell}{\sqrt{\phi_x}}}$$

$$f_1' = \frac{\eta_2}{h} + \frac{\eta_1}{\sqrt{\phi_y}} \; \frac{1}{\tanh \frac{H_\ell}{\sqrt{\phi_y}}}$$

$$f_2 = \frac{\eta_2}{h} + \frac{\eta_3}{\sqrt{\phi_x}} \; \frac{1}{\tanh \frac{H_f}{\sqrt{\phi_x}}}$$

$$f_2' = \frac{\eta_2}{h} + \frac{\eta_3}{\sqrt{\phi_y}} \; \frac{1}{\tanh \frac{H_f}{\sqrt{\phi_y}}} \tag{5.13}$$

$$f_3 = \frac{\eta_1}{\sqrt{\phi_x}} \; \frac{1}{\sinh \frac{H_\ell}{\sqrt{\phi_x}}}$$

$$f_3' = \frac{\eta_1}{\sqrt{\phi_y}} \; \frac{1}{\sinh \frac{H_\ell}{\sqrt{\phi_y}}}$$

$$f_4 = \frac{\eta_3}{\sqrt{\phi_x}} \; \frac{1}{\sinh \frac{H_f}{\sqrt{\phi_x}}}$$

$$f_4' = \frac{\eta_3}{\sqrt{\phi_y}} \; \frac{1}{\sinh \frac{H_f}{\sqrt{\phi_y}}}$$

$$g_1 = \frac{h}{2} + \sqrt{\phi_x}\ \tanh \frac{H_\ell}{2\sqrt{\phi_x}}$$

$$g_1' = \frac{h}{2} + \sqrt{\phi_y}\ \tanh \frac{H_\ell}{2\sqrt{\phi_y}}$$

$$g_2 = \frac{h}{2} + \sqrt{\phi_x}\ \tanh \frac{H_f}{2\sqrt{\phi_x}}$$

$$g_2' = \frac{h}{2} + \sqrt{\phi_y}\ \tanh \frac{H_f}{2\sqrt{\phi_y}}$$

Solving equations (5.12), the expressions for U_i, V_i, U_j and V_j can be obtained as follows :

$$U_i = \frac{1}{f_1\ f_2 - (\frac{n_2}{h})^2}\ [(U_1 f_2 f_3 + \frac{n_2}{h} f_4 U_2) - (g_1 f_2 + \frac{n_2}{h} g_2)\frac{\partial p}{\partial x}]$$

$$V_i = \frac{1}{f_1'\ f_2' - (\frac{n_2}{h})^2}\ [(V_1 f_2' f_3' + \frac{n_2}{h} f_4' V_2) - (g_1' f_2' + \frac{n_2}{h} g_2')\frac{\partial p}{\partial y}]$$

(5.14-a)

$$U_j = \frac{1}{f_1 f_2 - (\frac{n_2}{h})^2}\ [(U_2 f_1 f_4 + \frac{n_2}{h} f_3 U_1) - (g_2 f_1 + \frac{n_2}{h} g_1)\frac{\partial p}{\partial x}]$$

$$V_j = \frac{1}{f_1' f_2' - (\frac{n_2}{h})^2}\ [(V_2 f_1' f_4' + \frac{n_2}{h} f_3' V_1) - (g_2' f_1' + \frac{n_2}{h} g_1')\frac{\partial p}{\partial y}]$$

(5.14-b)

Further, in this case, the flow fluxes can be defined as follows :

$$Q_x = \int_{H_1}^{H_2} u\ dz = \int_{H_1}^{H_1+H_\ell} u_1\ dz + \int_{H_1+H_\ell}^{H_2-H_f} u_2\ dz + \int_{H_2-H_f}^{H_2} u_3\ dz \qquad (5.15)$$

$$Q_y = \int_{H_1}^{H_2} v\, dz = \int_{H_1}^{H_1+H_\ell} v_1\, dz + \int_{H_1+H_\ell}^{H_2-H_f} v_2\, dz + \int_{H_2-H_f}^{H_2} v_3\, dz \qquad (5.16)$$

After using the expressions for velocity in different regions appropriately, these equations give,

$$Q_x = U_1 \sqrt{\phi_x} \tanh \frac{H_\ell}{2\sqrt{\phi_x}} + U_2 \sqrt{\phi_x} \tanh \frac{H_f}{2\sqrt{\phi_x}} + U_i\, g_1 + U_j\, g_2$$

$$- [\frac{h^3}{12\eta_2} + \frac{\phi_x}{\eta_1} (H_\ell - 2\sqrt{\phi_x} \tanh \frac{H_\ell}{2\sqrt{\phi_x}})$$

$$+ \frac{\phi_x}{\eta_3} (H_f - 2\sqrt{\phi_x} \tanh \frac{H_f}{2\sqrt{\phi_x}})] \frac{\partial p}{\partial x} \qquad (5.15\text{-a})$$

$$Q_y = V_1 \sqrt{\phi_y} \tanh \frac{H_\ell}{2\sqrt{\phi_y}} + V_2 \sqrt{\phi_y} \tanh \frac{H_f}{2\sqrt{\phi_y}} + V_i\, g_1' + V_j\, g_2'$$

$$-[\frac{h^3}{12\ \ 2} + \frac{\phi_y}{\eta_1} (H_\ell - 2\sqrt{\phi_y} \tanh \frac{H_\ell}{2\sqrt{\phi_y}})$$

$$+ \frac{\phi_y}{\eta_3} (H_f - 2\sqrt{\phi_y} \tanh \frac{H_f}{2\sqrt{\phi_y}})] \frac{\partial p}{\partial y} \qquad (5.16\text{-b})$$

Integrating equation of continuity (5.4) with respect to z and taking the limits from $z = H_1$ to $z = H_2$, one gets

$$\int_{H_1}^{H_2} \frac{\partial u}{\partial x}\, dz + \int_{H_1}^{H_2} \frac{\partial v}{\partial y}\, dz + [w]_{H_1}^{H_2} = 0 \qquad (5.17)$$

Using, well known formula

$$\frac{\partial}{\partial x}\int_{H_1}^{H_2} f(x,y,z)\,dz = \int_{H_1}^{H_2}\frac{\partial}{\partial x} f(x,y,z)\,dz + \frac{\partial H_2}{\partial x}\,.\,f(x,y,H_2) - \frac{\partial H_1}{\partial x}\,.\,f(x,y,H_1) \tag{5.18}$$

equation (3.17) can be rewritten as follows :

$$\frac{\partial}{\partial x}\int_{H_1}^{H_2} u\,dz + \frac{\partial}{\partial y}\int_{H_1}^{H_2} v\,dz - U_2\frac{\partial H_2}{\partial x} + U_1\frac{\partial H_1}{\partial x} - V_2\frac{\partial H_2}{\partial y} + V_1\frac{\partial H_1}{\partial y} + [w]_{H_1}^{H_2} = 0 \tag{5.19}$$

Now on substituting the expressions of Q_x and Q_y in equation (5.19), we have

$$\frac{\partial}{\partial x}[\{\frac{h^3}{12\eta_2} + \frac{\phi_x}{\eta_1}(H_\ell - 2\sqrt{\phi_x}\tanh\frac{H_\ell}{2\sqrt{\phi_x}}) + \frac{\phi_x}{\eta_3}(H_f - 2\sqrt{\phi_x}\tanh\frac{H_f}{2\sqrt{\phi_x}})\}\frac{\partial p}{\partial x}]$$

$$+\frac{\partial}{\partial y}[\{\frac{h^3}{12\eta_2} + \frac{\phi_y}{\eta_1}(H_\ell - 2\sqrt{\phi_y}\tanh\frac{H}{2\sqrt{\phi_y}}) + \frac{\phi_y}{\eta_3}(H_f - 2\sqrt{\phi_y}\tanh\frac{H_f}{2\sqrt{\phi_y}})\}\frac{\partial p}{\partial y}]$$

$$= U_1\frac{\partial}{\partial x}[H_1 + \sqrt{\phi_x}\tanh\frac{H_\ell}{2\sqrt{\phi_x}}] - U_2\frac{\partial}{\partial x}[H_2 - \sqrt{\phi_x}\tanh\frac{H_f}{2\sqrt{\phi_x}}]$$

$$+ V_1\frac{\partial}{\partial y}[H_1 + \sqrt{\phi_y}\tanh\frac{H_\ell}{2\sqrt{\phi_y}}] - V_2\frac{\partial}{\partial y}[H_2 - \sqrt{\phi_y}\tanh\frac{H_f}{2\sqrt{\phi_y}}]$$

$$+\frac{\partial}{\partial x}(U_i\,g_1 + U_j\,g_2) + \frac{\partial}{\partial y}(V_i\,g_1' + V_j\,g_2') + [w]_{H_1}^{H_2} \tag{5.20}$$

This is a generalized form of Reynolds equation governing the pressure under the condition of mixed lubrication.

In the following, particular cases are derived from the generalized equation (5.20) and their one dimensional forms are studied.

i. SLIDER BEARING.

ii. HYDROSTATIC STEP BEARING.

iii. SQUEEZE FILM BEARING.

It is pointed out here that in the above cases, we have also studied the situation where the condition of mixed lubrication arise either in part of the bearing region or in whole of the bearing contact area. In the region where mixed lubrication exists, we have calculated the load capacity and the friction force only due to the hydrodynamic flow and the calculation of the components of these quantities which arise because of asperity contacts has not been included here. However, this can be easily done as suggested by Christensen [1972] and Tsao and Tong [1975] .

5.3 SLIDER BEARING UNDER MIXED LUBRICATION :

Consider the case of a slider bearing where the lower plate is smooth and moving with a constant velocity U. The upper plate forms the aforesaid interacting zone due to asperities through which the lubricant with viscosity η_1 flows. It may be considered that the region $0 \leq x \leq \ell_1$ may function under the condition of hydrodynamic lubrication and the region $\ell_1 \leq x \leq \ell$ is under mixed lubrication, Christensen [1972] .

By choosing,

$$H_1 = H_\ell = 0 \;,\; H_2 = h + H \;,\; \eta_1 = \eta_3$$

$$U_1 = U,\; U_2 = V_1 = V_2 = 0 \;,\; [w]_0^{h+H} = 0$$

equation (5.20) reduces to the following form in the region $0 \leq x \leq \ell_i$.

$$\frac{\partial}{\partial x}\left[F_x \frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[F_y \frac{\partial p}{\partial y}\right] = \frac{U}{2}\frac{\partial}{\partial x}\left[h + H\, G_x\right] \tag{5.21}$$

where

$$F_x = \frac{h^3}{12\eta_2} + \frac{H^3}{4\eta_1}\left[\frac{M_x - \tanh M_x}{M_x^3}\right] + \frac{hH^2}{4}\left[\frac{\alpha M_x + \tanh M_x}{M_x}\right]^2 \left[\frac{\tanh 2M_x}{2\eta_1 \;\alpha M_x + \eta_2 \tanh 2M_x}\right] \tag{5.22-a}$$

$$F_y = \frac{h^3}{12\eta_2} + \frac{H^3}{4\eta_1}\left[\frac{M_y - \tanh M_y}{M_y^3}\right] + \frac{hH^2}{4}\left[\frac{\alpha M_y + \tanh M_y}{M_y}\right]^2 \left[\frac{\tanh 2M_y}{2\eta_1 \;\alpha M_y + \eta_2 \tanh 2M_y}\right] \tag{5.22-b}$$

$$G_x = \left[\frac{\alpha M_x + \tanh M_x}{M_x}\right]\left[\frac{\eta_2 \tanh 2M_x}{2\eta_1 \;\alpha M_x + \eta_2 \tanh 2M_x}\right] \tag{5.22-c}$$

$$\alpha = \frac{h}{H} \;,\quad M_x = \frac{H}{2\sqrt{\phi_x}} \;,\quad M_y = \frac{H}{2\sqrt{\phi_y}}$$

and M_x, M_y may be called 'interference parameters' and they increase as the resistance to the flow increases.

The equation (5.21) determines the pressure in the region $0 \leq x \leq \ell_i$ where the viscosities in the layer $0 \leq y \leq h$ and $h \leq y \leq h+H$

are taken to be different due to the presence of surfactant in the lubricant. However, if this is not the case, $\eta_1 = \eta_2$ can be easily taken.

In the region $\ell_i \leq x \leq \ell_1$, the equation (5.21) determines the pressure with h = 0 and the functions F_x, G_x etc. are modified accordingly.

To see the effects of the parameter of the interacting zone on the characteristics of the bearing in a simplified form, the case of an infinite step slider bearing is considered where the step is located at $x = \ell_i$. In the region $0 \leq x \leq \ell_i$, $h = h_i$, it is considered that the bearing functions under hydrodynamic condition with interferance of the interacting net work zone while in the region $\ell_i \leq x \leq \ell$, $h = 0$, it functions under mixed lubrication with some flow through interacting net work zone corresponding to this region. The physical configuration is illustrated in Fig.(5.2).

In this case, Reynolds equation (5.21) reduces to

$$\frac{dp_i}{dx} = \text{const.} \qquad \begin{array}{l} i = 1 \text{ in region I} \\ i = 2 \text{ in region II} \end{array} \tag{5.23}$$

Since the pressure at the step p_s is continuous,

$$p_s = \ell_i \frac{dp_1}{dx} = -(\ell - \ell_i)\frac{dp_2}{dx} \tag{5.24}$$

Using the continuity of flow fluxes in the two regions and remembering equation (5.15-a), we can write

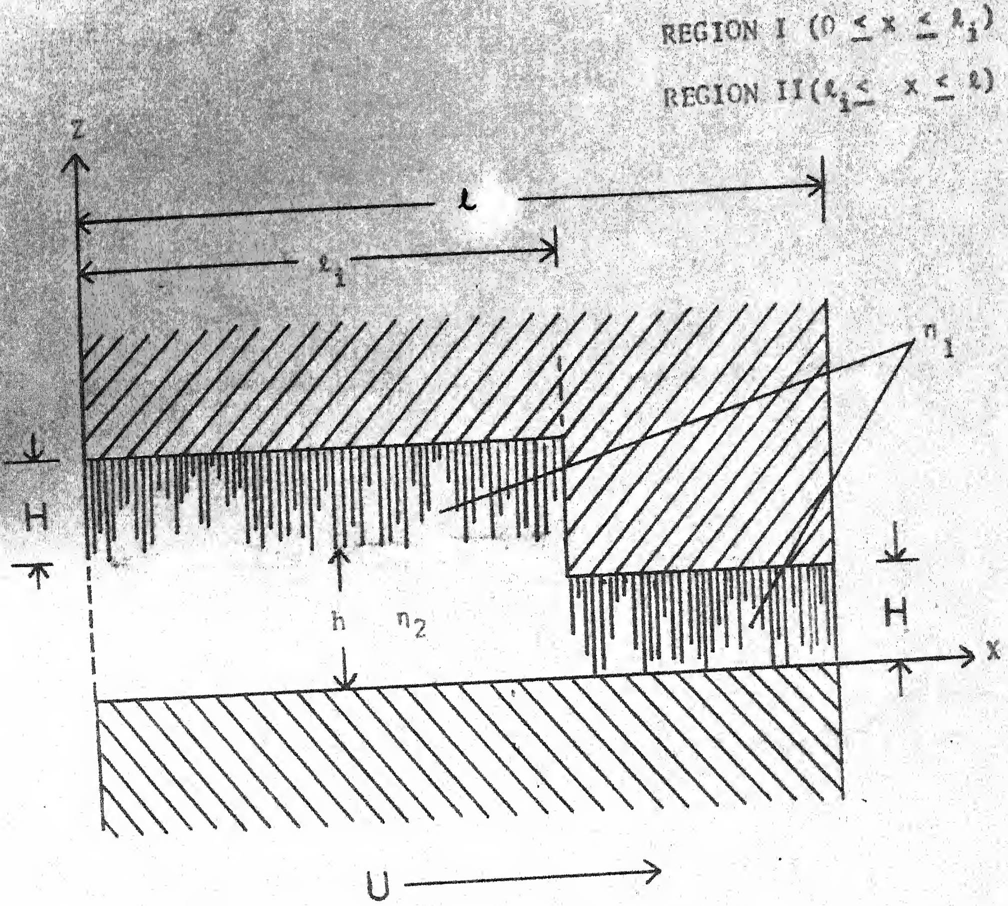

Fig. 5.2 STEP SLIDER BEARING UNDER MIXED LUBRICATION

$$Q = - F_x \frac{dp_1}{dx} + \frac{U}{2}(h + H\,G_x) \qquad (5.25)$$

$$= - F_o \frac{dp_1}{dx} + \frac{U}{2} H\,G_o$$

where F_x and G_x are defined in equations (5.22) and,

$$F_o = \frac{H^3}{4\eta_1}\left(\frac{M_x - \tanh M_x}{M_x^3}\right) \qquad (5.26)$$

$$G_o = \frac{\tanh M_x}{M_y} .$$

Solving equations (5.24) and (5.25), the following expressions for pressure gradient can be obtained.

$$\frac{dp_1}{dx} = \frac{U(\ell - \ell_i)}{2} \frac{h - H(G_o - G_x)}{(\ell - \ell_i) F_x + \ell_i F_o} , \quad 0 \le x \le \ell_i \qquad (5.27)$$

and

$$\frac{dp_2}{dx} = - \frac{U \ell_i}{2} \frac{h - H(G_o - G_x)}{(\ell - \ell_i) F_x + \ell_i F_o} , \quad \ell_i \le x \le \ell \qquad (5.28)$$

Using the boundary conditions

$$p_1 = 0 \text{ at } x = 0$$

$$p_1 = p_2 \text{ at } x = \ell_i \qquad (5.29)$$

$$p_2 = 0 \text{ at } x = \ell$$

the expression for the load capacity can be evaluated as follows :

$$W = \int_0^{\ell_i} p_1\,dx + \int_{\ell_i}^{\ell} p_2\,dx$$

$$= - \int_0^{\ell_i} x\left(\frac{dp_1}{dx}\right) dx - \int_{\ell_i}^{\ell} x\left(\frac{dp_2}{dx}\right) dx .$$

On using equations (5.27) and (5.28), this gives,

$$W = \frac{U \ell\ell_i (\ell - \ell_i)}{4} \left[\frac{h - H(G_o - G_x)}{(\ell - \ell_i) F_x + \ell_i F_o} \right] \tag{5.30}$$

Following the usual procedure, shear stress at the moving surface can be obtained as follows :

$$\tau_1 = \frac{1}{2}(h + HG_x)\frac{dp_1}{dx} + \frac{1}{H} \frac{2\eta_1 \eta_2 U M_x}{2\alpha\eta_1 M_x + \eta_2 \tanh 2M_x},$$

$$\text{for } 0 \leq x \leq \ell_i \tag{5.31}$$

$$\tau_2 = \frac{H}{2} G_o \frac{dp_2}{dx} + \frac{\eta_1 U}{H} \left(\frac{2M_x}{\tanh 2M_x}\right), \text{ for } \ell_i \leq x \leq \ell \tag{5.32}$$

Now the force of friction at the moving surface can be obtained by integrating the above expressions over the interval $0 \leq x \leq \ell$,

i.e.

$$F = \int_0^{\ell_i} \tau_1 \, dx + \int_{\ell_i}^{\ell} \tau_2 \, dx$$

which gives,

$$F = \frac{U \ell_i (\ell - \ell_i)}{4} \frac{[h - H(G_o - G_x)]^2}{(\ell - \ell_i)F_x + \ell_i F_o}$$
$$+ \frac{\eta_1 U (\ell - \ell_i)}{H} \left(\frac{2M_x}{\tanh 2M_x}\right)$$
$$+ \frac{U \ell_i}{H} \cdot \frac{1}{\frac{\alpha}{\eta_2} + \frac{1}{\eta_1}\left(\frac{\tanh 2M_x}{2M_x}\right)} \tag{5.33}$$

To see the various effects of interference parameter on the load capacity and the force of friction, it is noted from the

appendix that the function $[h - H(G_o - G_x)]$ is positive and increases as M_x $(= \frac{H}{2\sqrt{\phi_x}})$ increases. Further, the functions, F_x, F_o, $\frac{\tanh 2M_x}{2M_x}$ decrease as M_x increases.

Then, it can be seen from equations (5.30) and (5.33) that the load capacity and the friction force increase as M_x increases. Since the increase in M_x implies more resistance (interference) to the flow, it shows that the load capacity and friction force increase as the interference due to asperities increases. It can also be noted from equations (5.30) and (5.33) that the load capacity and friction force increase as η_1 or η_2 increase.

If there is no interference of the interacting net-work zone in the hydrodynamic region $0 \leq x \leq \ell_i$ while in the region $\ell_i \leq x \leq \ell$ (where $h = 0$), the condition of mixed lubrication exist as shown in Fig. (5.3), the load capacity and friction force can be calculated by following the above procedure and we can obtain the following expressions

$$W = \frac{U\,\ell\ell_i(\ell - \ell_i)}{4} \; \frac{(h - H\,G_o)}{(\ell - \ell_i)\,F_{x1} + \ell_i\,F_o} \tag{5.34}$$

$$F = \frac{U\,\ell_i(\ell - \ell_i)}{4} \; \frac{(h - H\,G_o)^2}{(\ell - \ell_i)\,F_{x1} + \ell_i\,F_o} + U\left[\frac{\eta_2\,\ell_i}{h} + \frac{\eta_1(\ell - \ell_i)}{H}\left(\frac{2M_x}{\tanh 2M_x}\right)\right] \tag{5.35}$$

where

$$F_{x1} = \frac{h^3}{12\eta_2}$$

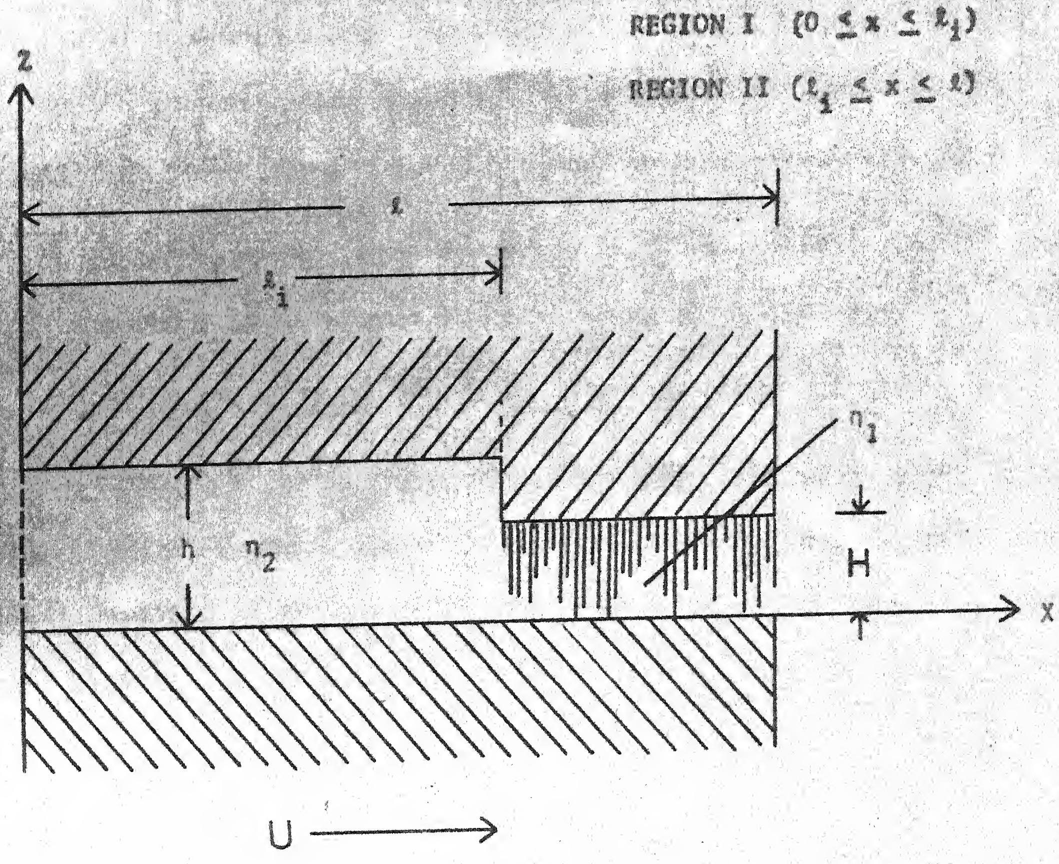

Fig. 5.3 STEP SLIDER BEARING UNDER MIXED LUBRICATION
(No interacting zone in $0 \leq x \leq \ell_1$)

Again, since $(h - H\,G_o)$ is an increasing function of M_x and F_o decreases as M_x increases, it can be seen from equations (5.34) and (5.35) that the load capacity and friction force increase with M_x as in the previous case which is due to higher resistance to the fluid flow through interacting net-work zone in region II, $(\ell_i \leq x \leq \ell)$.

It may also seen from equations (5.34) and (5.35) that the load capacity and friction force increase as η_1 or η_2 increases

5.4 HYDROSTATIC STEP BEARING UNDER MIXED LUBRICATION

Consider the case of hydrostatic step bearing as shown in Fig. (5.4) where the upper plates forms an interacting net work zone due to asperities through which the lubricant with viscosity η_1 flows. In this case, Reynolds equation can be obtained from equation (5.21) as follows :

$$\frac{d}{dr}\left[r\, F_i \frac{dp}{dr} \right] = 0 \qquad \begin{array}{l} i = 1 \text{ for } r_i \leq r \leq r_s \\ i = 2 \text{ for } r_s \leq r \leq r_e \end{array} \tag{5.36}$$

where,

$$F_1 = \frac{h^3}{12\eta_2} + \frac{H^3}{4\eta_1}\left[\frac{M - \tanh M}{M^3}\right] + \frac{hH^2}{4}\left[\frac{\alpha M + \tanh M}{M}\right]^2 \left[\frac{\tanh 2M}{2\eta_1\, \alpha M + \eta_2 \tanh 2M}\right] \tag{5.37-a}$$

$$F_2 = \frac{H^3}{4\eta_1}\; \frac{M - \tanh M}{M^3} \tag{5.37-b}$$

$$M = \frac{H}{2\sqrt{\phi}} \quad , \quad \alpha = \frac{h}{H} \tag{5.37-c}$$

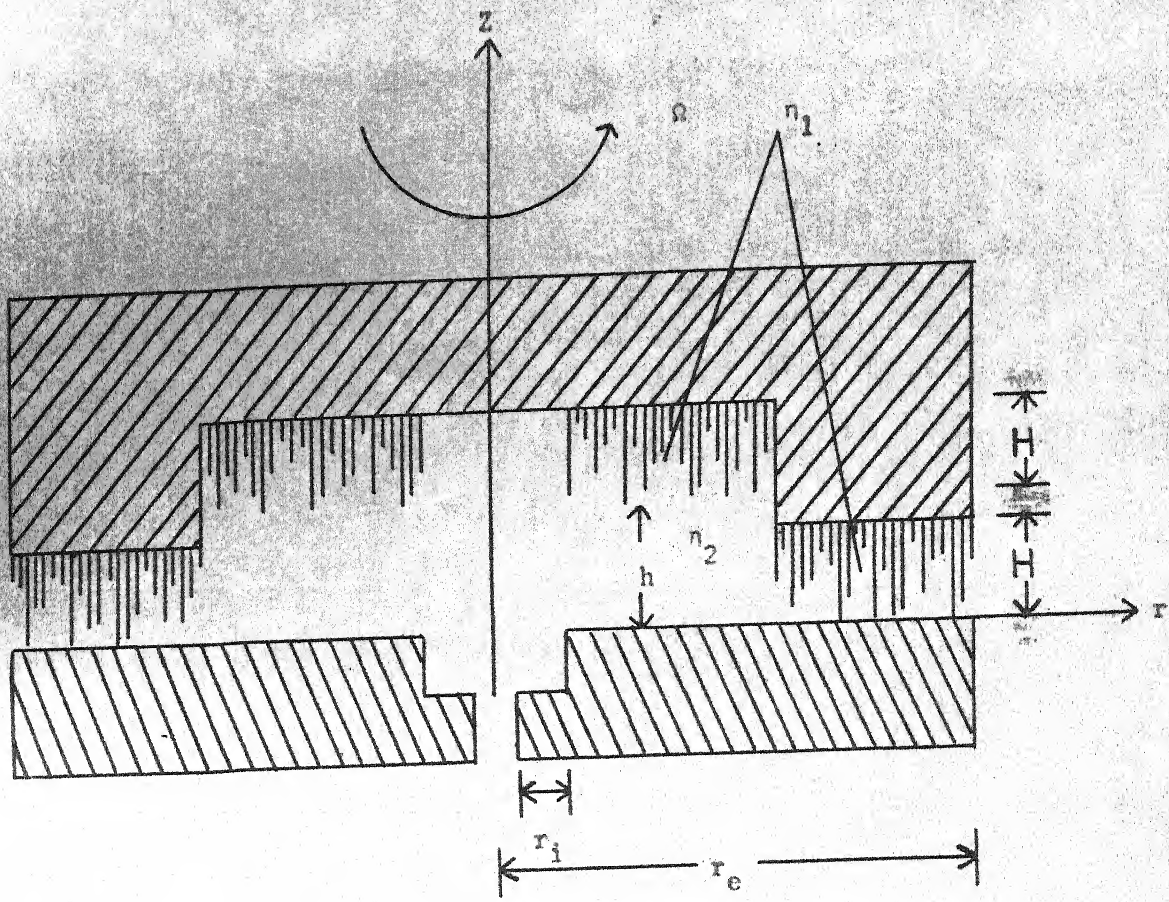

Fig. 5.4. HYDROSTATIC STEP BEARING UNDER MIXED LUBRICATION

Integrating equation (5.36) and using the definition of flow flux (the flux is constant in this case), we have,

$$Q = -2\pi r F_1 \frac{dp_1}{dr} \tag{5.38}$$
$$= -2\pi r F_2 \frac{dp_2}{dr}$$

Integrating equations (5.38) with respect to r and using the boundary conditions

$$p_1 = p_i \text{ at } r = r_i$$

$$p_1 = p_2 = p_s \text{ at } r = r_s$$

$$p_2 = 0 \text{ at } r = r_e$$

the expressions for pressure can be obtained as follows :

$$p_1 = p_i - \frac{Q}{2\pi F_1} \ln \frac{r}{r_i} \tag{5.39}$$

$$p_2 = -\frac{Q}{2\pi F_2} \ln \frac{r}{r_e}$$

$$p_s = p_i - \frac{Q}{2\pi F_1} \ln \frac{r_s}{r_i} = -\frac{Q}{2\pi F_2} \ln \frac{r_s}{r_e}$$

By using equations (5.38) and (5.39) the expression for flux can be written as,

$$Q = \frac{2\pi p_i}{\frac{1}{F_1} \ln \frac{r_s}{r_i} + \frac{1}{F_2} \ln \frac{r_e}{r_s}} \tag{5.40}$$

The load capacity in this case can be evaluated by

$$W = \pi r_i^2 p_i + \int_{r_i}^{r_s} 2\pi r p_1 dr + \int_{r_s}^{r_e} 2\pi r p_2 dr \qquad (5.41)$$

which on using equation (5.39) gives,

$$W = \frac{Q}{4} \left[\frac{(r_s^2 - r_i^2)}{F_1} + \frac{(r_e^2 - r_s^2)}{F_2} \right] \qquad (5.42)$$

Since functions F_1 and F_2 are similar to the function F_x and F_o as defined in the equations (5.22) and equation (5.26), and are also decreasing function of M, it can be seen from equation (5.42) that for a constant flow flux the load capacity increases as M increases i.e. as the interference to the flow increases.

It may also be noted from equation (5.40) that for a given inlet pressure the flux decreases as the interference parameter increases.

Further, as F_1 and F_2 also decreases as η_1 or η_2 increases, it is concluded that the load capacity increases as η_1 or η_2 increases for given flux but the flow flux decreases for given inlet pressure.

5.5 SQUEEZE FILM BEARING

Consider the case of squeeze film through three layers which may be applicable to synovial joint functioning under the condition of standing, jumping etc. as shown in Fig. (5.5). Under hydrodynamic condition of the thickness of the synovial fluid film is non zero but when the joint function under the condition of mixed lubrication, $h = 0$. This condition arises at the fag end of jumping or when falling

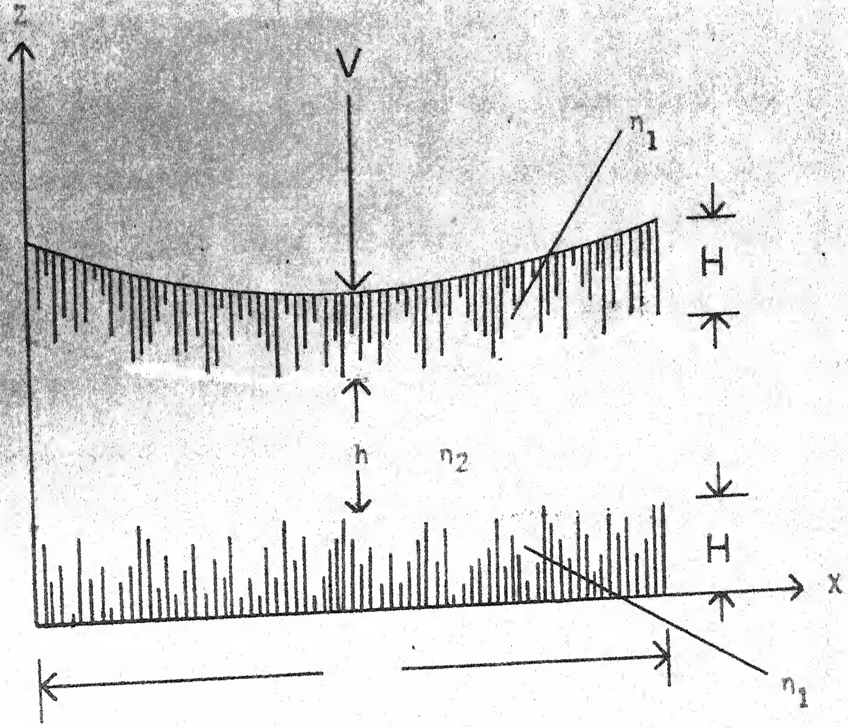

Fig. 5.8 SYNOVIAL JOINT

accidently and creating impact loading condition. In the last condition, the cartilage may get fractured and ruptured and thus affecting the interference parameter M_x. In fact, due to the fracture, the resistance to flow through the interacting net-work zones would decrease and M_x would decrease.

To see the effect of M_x, on the characteristics of joint, very simply and clarly it can be assumed that the lower surface of the cartilage is plane while the upper surface may be plane or curved. Further, it is assumed that under impact loading or near impact loading condition, since the cartilages are elastic, the contact zone has aconstant film-thickness.

Keep the above in view and assuming that the thickness of middle layer is small but not zero, the equation (5.20) reduces to the following one-dimensional form, as applicable to the present situation, ($H_1 = 0$, $H_\ell = H_f = H$, $U_1 = U_2 = 0$)

$$\frac{d}{dx}\left[F_{x2}\frac{dp}{dx}\right] = -V \tag{5.43}$$

where

$$F_{x2} = \frac{h^3}{12n_2} + \frac{H^3}{2n_1}\left[\left(\frac{M_x - \tanh M_x}{M_x^3}\right) + \left(\frac{\tanh 2M_x}{2M_x}\right)\left(\frac{\alpha M_x + \tanh M_x}{M_x}\right)^2\right] \tag{5.44}$$

Integrating equation (5.43) and using boundary conditions

$$\frac{dp}{dx} = 0 \text{ at } x = \frac{\ell}{2} \tag{5.45}$$

we have,

$$\frac{dp}{dx} = - \frac{V}{F_{x2}} \left(x - \frac{\ell}{2}\right) \tag{5.46}$$

By considering $p = 0$ at $x = 0$ and $x = \ell$, the load capacity of the joint can be calculated as follows :

$$W = \int_0^{\ell} \left(-x \frac{dp}{dx}\right) dx = V \int_0^{\ell} \frac{x(x - \frac{\ell}{2})}{F_{x2}} dx \tag{5.47}$$

The time of squeezing T for a constant load W can be obtained as follows, by writing

$$V = - \frac{d}{dt} (h + H) = - \frac{dh}{dt}$$

$$T = \frac{1}{W} \int_{h_f}^{h_i} I(h)\, dh \tag{5.48}$$

where H is assumed constant and

$$I(h) = \int_0^{\ell} \frac{x\, (x - \frac{\ell}{2})}{F_{x2}} dx \tag{5.49}$$

From the expression of F_{x2} in equation (5.44) and the appendix, it can be seen that F_{x2} decreases as M_x, η_1 or η_2 increase. Thus in this case also it may be noted that the load capacity increases due to decrease in F_{x2}. Hence, the load capacity and the time of squeezing decrease as M_x, η_1 , η_2 decrease. The decrease in M_x may be caused due to fractured cartilage while decrease in η_1 or η_2 may be caused because of high shear rate under impact loading conditions. Also, in the above process a situation can arise where η_2 , the viscosity of the synovial fluid film of thickness h may impact be increased because of the presence

of hyaluronic acid molecules due to boosted lubrication, Dowson, et. al. [1970] . This increase in η_2 might compensate its decrease due to high shear rate under impact loading condition.

Thus, it may be concluded that the fractured joint functions less effectively than the normal joint. But this might be compensated because of the novel structure of the joint provided by nature even under severe conditions.

Under the condition of severe impact loading, the two interacting surface net-work zones of the cartilage would be in contact (h = 0) over the whole region and the two surfaces are almost parallel in this region [see Fig. (5.6)]. In this case, function F_x reduces to the following form,

$$F_{xo} = (F_{x2})_{h=0} = \frac{H^3}{2\eta_1} \left[\left(\frac{M_x - \tanh M_x}{M_x^3} \right) + \left(\frac{\tanh 2M_x}{2M_x} \right) \left(\frac{\alpha M_x + \tanh M_x}{M_x} \right)^2 \right] \quad (5.50)$$

As before, the expressions for load capacity, in this case can be obtained as follows :

$$W = V \int_0^{\ell} \frac{x(x - \frac{\ell}{2})}{F_{xo}} = \frac{V\ell^3}{12\ F_{xo}} \quad (5.51)$$

Since the cartilage are elastic, the thickness of interacting net-work zones can decrease because of loading and squeezing can also take place even in this case. Then by considering

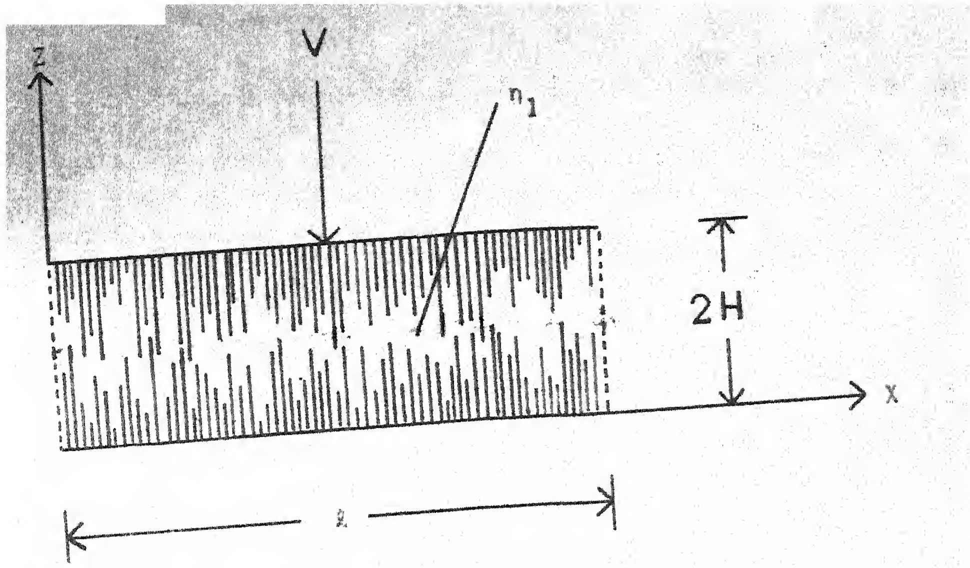

Fig. 5.6 : SYNOVIAL JOINT (CARTILAGES ARE IN CONTACT)

$V = -\frac{d}{dt}(2H) = -2\frac{dH}{dt}$, the time of squeezing for a constant load W can be obtained as follows :

$$T = \frac{\ell^3}{6W}\int_{2H_f}^{2H_i}\frac{dH}{F_{xo}} \tag{5.52}$$

Again, since F_{xo} increases as M_x or η_1 decrease, it can be seen from the equations (5.51) and (5.52) that the load capacity and time of squeezing decrease as M_x or η_1 decrease due to fractured cartilage under the condition of impact loading.

Thus, a person with a fractured cartilage may not able to stand and support his body load due to continuous contact of the cartilage surface and quick squeezing of the fluid in the interference zones.

5.6 CONCLUSIONS

A new theory of mixed lubrication is presented by considering a modified forms of equations of motion in interacting net-work zones formed by surface asperities and a generalized form of Reynolds equation is derived.

As particular cases, step slider bearing, hydrostatic bearing have been studied. It is shown that the load capacity and friction force increase as the interference to the flow increases or as viscosities of layers increase.

In the case of synovial joint, it has been pointed out that the load capacity and time of squeezing decrease due to fractured joint under impact loading condition. However, this may be compensated by boosting effects.

APPENDIX

The derivative of the function $\frac{M_x - \tanh M_x}{M_x^3}$ is given as follows :

$$\frac{d}{dM_x}\left(\frac{M_x - \tanh M_x}{M_x^3}\right) = -\frac{\Psi(M_x)}{M_x^4} \tag{5.53}$$

where

$$\Psi(M_x) = 3M_x - M_x \tanh^2 M_x - 3 \tanh M_x \tag{5.54}$$

Since, $\Psi(0) = 0$ and

$$\frac{d}{dM_x}\Psi(M_x) = 2 \tanh M_x(\tanh M_x - M_x \operatorname{sech}^2 M_x) \tag{5.55}$$

is positive for all values of $M_x > 0$. Hence $\Psi(M_x)$ is an increasing function of M_x and positive for all values of $M_x > 0$. Then it can be concluded from equation (5.53), that $\frac{M_x - \tanh M_x}{M_x^3}$ is a decreasing function of M_x for all values of $M_x > 0$.

In a similar manner, it can also be shown that $\frac{\tanh M_x}{M_x}$ is also a decreasing function of M_x for all values of $M_x > 0$.

Keeping these in mind, it can be noted from equations (5.22-a), (5.26), (5.37-a), (5.37-b), (5.44) and (5.50) that the functions F_x, F_o, F_1, F_2, F_{x2} and F_{xo} decrease as M_x increases. It can also be noted that these functions also decrease as η_1 or η_2 increases.

Now consider the function,

$$G_x = \left[\alpha + \frac{\tanh M_x}{M_x}\right]\left[\frac{\eta_2}{\eta_2 + \alpha\eta_1 \frac{2M_x}{\tanh 2M_x}}\right] \tag{5.56}$$

and

$$G_o = \frac{\tanh M_x}{M_x} \qquad (5.57)$$

As before it can be seen from the above equations (5.56) (5.57) that G_x and G_o are decreasing functions of M_x for all values of $M_x > 0$.

Further as M_x tends to zero, G_x and G_o tend to $(\alpha+1)\frac{\eta_2}{\eta_2 + \alpha\eta_1}$ and one respectively. Again as M_x tends to ∞, G_x or G_o tends to zero. Also,

$$\frac{(\alpha+1)\,\eta_2}{\eta_2 + \alpha\eta_1} = \frac{(\alpha+1)}{\alpha\frac{\eta_1}{\eta_2} + 1} \begin{cases} < 1 & \text{for } \eta_1 > \eta_2 \\ = 1 & \text{for } \eta_1 = \eta_2 \end{cases} \qquad (5.58)$$

Since G_x and G_o are decreasing functions of M_x, $(G_o - G_x)$ is positive for $\eta_1 > \eta_2$, and decreases as M_x increases for all values of $M_x > 0$.

Keeping these in view and noting that, $h > H$, in the case of slider bearing[see Figs. (5.2) and (5.3)] and $(G_o - G_x) < 1$, it is concluded that the functions $\{h - H(G_o - G_x)\}$, $\{h - HG_o\}$ increase as M_x increases.

LIST OF NOTATIONS

F	-	friction force
h	-	film-thickness
$H, H\ , H_f$	-	thickness of the interference zones
H_1, H_2	-	distance between the surfaces and xy plane
i	-	denoted for ith interface
j	-	denoted for jth interface
ℓ	-	bearing length
ℓ_i	-	step position
p, p_1, p_2	-	fluid film pressure
p_i	-	inlet pressure
p_s	-	step pressure
Q, Q_x, Q_y	-	flux
r	-	radial coordinate
r_i	-	inlet radious
r_e	-	outlet radious
r_s	-	step position
T	-	time of squeezing
u, v, w	-	velocities of fluid in x, y and z directions
u_1, v_1	-	velocities of fluid in x and y directions in the region I
u_2, v_2	-	velocities of fluid in x and y directions in the region II
u_3, v_3	-	velocities of fluid in x and y directions in the region III
U_1, V_1	-	velocities of the surface at $z = H_1$ along x and y directions.

U_2, V_2	-	velocities of the surface at $z = H_2$ along x and y directions.
U_i, V_i	-	velocities at $z = H_1 + H_\ell$ along x and y directions
U_j, V_j	-	velocities at $z = H_2 - H_f$ along x and y directions
V	-	squeeze velocity
W	-	load capacity
x,y,z	-	rectangular coordinate system
η_1, η_2, η_3	-	viscosities of lubricant in the regions I, II and III

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